Large-time Behavior of Solutions for the 1D Viscous Heat-Conducting Gas with Radiation: the Pure Scattering Case

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Outline of This Talk

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- Some previous works
- The main results
- Global existence of solutions
- Large-time behavior of solutions
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- References
1 Introduction to the model

In this talk we are concerned with the following model of the viscous gas with radiation: the pure scattering case

\[
\begin{align*}
\eta_t &= v_x, \\
v_t &= \sigma_x - \eta (S_F) R, \\
(e + \frac{1}{2}v^2) &= (\sigma v - Q)_x, \\
I_t + \eta^{-1}(\omega v - v)I_x &= c\eta S.
\end{align*}
\]

(1.1)

\(\eta = \frac{1}{\rho}\) — specific volume, \(v\) — velocity,

\(\theta\) — temperature, \(I(x, t; \omega, \nu)\) — radiative intensity,

\(\nu \in R^+ = (0, +\infty)\) — radiation frequency variable, \(\omega \in S^1 := [-1, 1]\) — angular variable,

\(\sigma = -p + \frac{\mu(\eta)v_x}{\eta}\) — stress, \(Q = -\kappa \frac{\theta_x}{\eta}\) — heat flux,

\(\kappa(\eta, \theta)\) — heat conductivity, \(\mu(\eta)\) — viscosity coefficient, \(S(x, t; \nu, \omega)\) — the source term,

\((S_F)_R = \frac{1}{c} \int_{S^1} \int_{0}^{\infty} \omega S d\nu d\omega\) — radiative force, \(c\) — velocity of light.
We consider a typical initial boundary value problem for (1.1) in the domain $Q := \Omega \times [0, +\infty) = (0, 1) \times [0, +\infty)$ under the Dirichlet-Neumann boundary conditions for the fluid unknowns

$$v(0, t) = v(1, t) = 0, \quad Q(0, t) = Q(1, t) = 0, \quad \forall \ t \geq 0,$$

and transparent boundary conditions for the radiative intensity

$$\begin{aligned}
I(0, t; \nu, \omega) &= 0 \quad \text{for} \ \omega \in (0, 1), \quad \forall \ t \geq 0, \\
I(1, t; \nu, \omega) &= 0 \quad \text{for} \ \omega \in (-1, 0), \quad \forall \ t \geq 0,
\end{aligned}$$

(1.2)

and initial conditions

$$\eta(x, 0) = \eta_0(x), \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x) \quad \text{on} \ \Omega,$$

and

$$I(x, 0; \nu, \omega) = I_0(x; \nu, \omega) \quad \text{on} \ \Omega \times R^+ \times S^1.$$
2 Some previous works

Recently, Ducomet and Nečasová [5,6,7,8] derived and studied the one-dimensional radiative fluid, which can be rewritten as follows

\[
\begin{cases}
\eta_t = v_x, \\
v_t = \sigma_x - \eta(S_F)_R, \\
(e + \frac{1}{2}v^2) = (\sigma v - Q)_x - \eta(S_E)_R, \quad (S_E)_R := \int_{S_1} \int_0^\infty Sd\nu d\omega, \\
I_t + \eta^{-1}(\omega - v)I_x = c\eta S.
\end{cases}
\]  

(2.1)

- In [6], Ducomet and Nečasová studied the system (2.1) with Dirichlet–Neumann boundary conditions for \( v \) and \( q \) and transparent boundary conditions for the radiative intensity:

\[
I|_{x=0} = 0 \text{ for } \omega \in (0, 1); \quad I|_{x=M} = 0 \text{ for } \omega \in (-1, 0).
\]

They took the source term

\[
S(x, t; \nu, \omega) = \sigma_a(\nu, \omega; \eta, \theta)[B(\nu; \theta) - I(x, t; \nu, \omega)] + \sigma_s(\nu; \eta, \theta)[\tilde{I}(x, t; \nu) - I(x, t; \nu, \omega)],
\]

where \( \tilde{I}(x, t; \nu) = \frac{1}{2} \int_1^{-1} I(x, t; \nu, \omega) d\omega \). Under suitable assumptions and \( q \geq r + 1 \), they proved the existence and uniqueness of weak solutions. But all the estimates depended on any given time \( T > 0 \). So they didn’t show the large-time behavior of problem (2.1).
• In [7], Ducomet and Nečasová studied the system (2.1) with Dirichlet–Neumann boundary conditions for

\[ v|_{x=0} = v|_{x=M} = 0, \quad \theta|_{x=0} = \theta_0, \quad q|_{x=M} = 0 \]

and transparent boundary conditions for the radiative intensity:

\[ I|_{x=0} = I_b \quad \text{for } \omega \in (0, 1); \quad I|_{x=M} = I_b \quad \text{for } \omega \in (-1, 0). \]

Under suitable hypotheses on the transport coefficients and adapted boundary conditions, they proved that the unique strong solution of this problem converges toward a well-determined equilibrium state at exponential rate.
In [5], Ducomet and Nečasová studied an “infrarelativistic” model, that is,

\[
\begin{align*}
\eta_t &= v_x, \\
v_t &= \sigma_x, \\
(e + \frac{1}{2}v^2) &= (\sigma v - Q)_x - \eta(S_E)_R, \\
\omega I_x &= \eta S.
\end{align*}
\]

Under suitable assumptions, they established the global existence of solutions with Dirichlet–Neumann boundary conditions for \( v \) and \( Q \) and transparent boundary conditions for the radiative intensity:

\[ I|_{x=0} = 0 \text{ for } \omega \in (0, 1); \quad I|_{x=M} = 0 \text{ for } \omega \in (-1, 0). \]

But the estimates obtained there depend on any given time \( T \). Moreover, all the estimates they obtained only were valid for \( q \geq 2r + 1 \).
• In [23], Qin, Feng and Zhang studied the system (2.2) and established the uniform-in-time estimates of \((\eta(t), v(t), \theta(t), \mathcal{I}(t) = \int_0^\infty \int_{S^1} I(x, t; \nu, \omega) d\omega d\nu)\) in \(\mathcal{H}_i = H^i(0, 1) \times H^i_0(0, 1) \times H^i(0, 1) \times H^{i+1}(0, 1), \ (i = 1, 2)\), which hold for \(q \geq r + 1\).

• In [10], Feng, Qin and Zhang continued to study the “infrarelativistic” model and obtained global existence of solutions and asymptotic behavior in \(\mathcal{H} = H^4(0, 1) \times H^4_0(0, 1) \times H^4(0, 1) \times H^5(0, 1)\) based on the results in [23].
When \((S_E)_R = 0\) in (2.1), that is,

\[
\begin{cases}
\eta_t = v_x, \\
v_t = \sigma_x - \eta(S_F)_R, \\
(e + \frac{1}{2}v^2) = (\sigma v - Q)_x, \\
\omega I_t + \eta^{-1}(c\omega - v)I_x = cS.
\end{cases}
\]  

(2.3)

Ducomet and Nečasová [8] took the source term in the last equation, including only scattering processes and neglecting absorption and emission phenomena

\[
S(x, t; \nu, \omega) = \sigma_s(\nu; \eta, \theta) \left[ \tilde{I}(x, t; \nu) - I(x, t; \nu, \omega) \right]
\]

and obtained the solutions \((\eta(t), v(t), \theta(t)) \in H^1(0, M)\) and \(\mathcal{I}(t) \in L^2(0, M)\) with \(\mathcal{I}(x, t) := \int_0^\infty \int_{S^1} I(x, t; \nu, \omega) d\omega d\nu\). Moreover, they proved the large-time behavior of solutions in \((H^1(0, M))^3 \times L^2(0, M)\).
Unfortunately, under the assumption on $\sigma_s$ in [8], they can not get the global existence and large-time behavior of solutions to the system in a much higher regular space.

In this work, under some new assumption on $\sigma_s$, we establish the global existence and large-time behavior of solutions to the system in $\mathcal{H}_2$ and $\mathcal{H}_3$ (see in section 3). Moreover, the global existence and large-time behavior of solutions $(\eta(t), v(t), \theta(t), I(t))$ in $\mathcal{H}_1$ are still valid under our assumption.
3 The main results

3.1. Some basic assumptions

Firstly, we define

\[ \mathcal{H}_1 = \left\{ (\eta, v, \theta, I) \in H^1(0, 1) \times H^1_0(0, 1) \times H^1(0, 1) \times L^2((0, 1) \times (-1, 1) \times R^+) : \right. \\
\quad \eta(x) > 0, \ \theta(x) > 0, \ x \in [0, 1], \ v|_{x=0,1} = 0, \ I|_{x=0} = 0 \text{ for } \omega \in (0, 1), \\
\quad I|_{x=1} = 0 \text{ for } \omega \in (-1, 0) \left. \right\}, \]

\[ \mathcal{H}_2 = \left\{ (\eta, v, \theta, I) \in H^2(0, 1) \times H^2_0(0, 1) \times H^2(0, 1) \times H^1((0, 1) \times (-1, 1) \times R^+) : \right. \\
\quad \eta(x) > 0, \ \theta(x) > 0, \ x \in [0, 1], \ v|_{x=0,1} = 0, \ \theta_x|_{x=0,1} = 0, \\
\quad I|_{x=0} = 0 \text{ for } \omega \in (0, 1), \ I|_{x=1} = 0 \text{ for } \omega \in (-1, 0) \left. \right\}, \]

\[ \mathcal{H}_3 = \left\{ (\eta, v, \theta, I) \in H^4(0, 1) \times H^4_0(0, 1) \times H^4(0, 1) \times H^3((0, 1) \times (-1, 1) \times R^+) : \right. \\
\quad \eta(x) > 0, \ \theta(x) > 0, \ x \in [0, 1], \ v|_{x=0,1} = 0, \ \theta_x|_{x=0,1} = 0, \\
\quad I|_{x=0} = 0 \text{ for } \omega \in (0, 1), \ I|_{x=1} = 0 \text{ for } \omega \in (-1, 0) \left. \right\}. \]
Then we assume $e, p, \sigma$ and $\kappa$ are fourth continuously differential on $0 < \eta < +\infty$ and $0 \leq \theta < +\infty$, and we suppose the following growth conditions:

$$
\begin{align*}
    e(\eta, 0) & \geq 0, \quad c_1(1 + \theta^r) \leq e_\theta(\eta, \theta) \leq C_1(1 + \theta^r), \\
    -c_2\eta^{-2}(1 + \theta^{1+r}) & \leq p_\eta(\eta, \theta) \leq -C_2\eta^{-2}(1 + \theta^{1+r}), \\
    |p_\theta(\eta, \theta)| & \leq C_3\eta^{-1}(1 + \theta^r), \\
    c_4(1 + \theta^{1+r}) & \leq \eta p_\eta(\eta, \theta) \leq C_4(1 + \theta^{1+r}), \quad p_\eta(\eta, \theta_0) \leq 0, \\
    0 & \leq p(\eta, \theta) \leq C_5(1 + \theta^{1+r}), \\
    c_6(1 + \theta^q) & \leq \kappa(\eta, \theta) \leq C_6(1 + \theta^q), \\
    |\kappa_\eta(\eta, \theta)| + |\kappa_{\eta\eta}(\eta, \theta)| & \leq C_7(1 + \theta^q), \\
    0 & < \sigma_s(\nu; \eta, \theta) \leq C_8|\theta - \overline{\theta}|^{\alpha}\kappa(\nu),
\end{align*}
$$

and

$$
\begin{align*}
    (|(\sigma_s)_\eta| + |(\sigma_s)_\theta|)(\nu; \eta, \theta) & \leq C_9l(\nu), \\
    (|(\sigma_s)_\theta\theta| + |(\sigma_s)_\eta\theta| + |(\sigma_s)_\eta\eta| + |(\sigma_s)_\theta\theta\theta| + |(\sigma_s)_\eta\eta\eta| \\
     + |(\sigma_s)_\eta\theta\theta| + |(\sigma_s)_\eta\eta\theta|)(\nu; \eta, \theta) & \leq C_{10}l(\nu),
\end{align*}
$$

and

$$
\begin{align*}
    (|(\sigma_s)_\eta| + |(\sigma_s)_\theta|)(\nu; \eta, \theta) & \leq C_9l(\nu), \\
    (|(\sigma_s)_\theta\theta| + |(\sigma_s)_\eta\theta| + |(\sigma_s)_\eta\eta| + |(\sigma_s)_\theta\theta\theta| + |(\sigma_s)_\eta\eta\eta| \\
     + |(\sigma_s)_\eta\theta\theta| + |(\sigma_s)_\eta\eta\theta|)(\nu; \eta, \theta) & \leq C_{10}l(\nu),
\end{align*}
$$
where \( \bar{\theta} \) is the static solution of (1.1) given in the following proposition, the numbers \( c_i, C_j, (i = 1, \ldots, 6, j = 1, \ldots, 10) \) are positive constants and the nonnegative functions \( k, l \) are such that

\[
k, l \in L^{1+\gamma}(R^+) \cap L^\infty(R^+), \quad \gamma \geq 0,
\]

and \( r, \alpha, q \) satisfy

\[
\frac{2}{3} \leq r \leq \frac{7}{2}, \quad 8 - r \leq q \leq \frac{13r}{2} + 3, \quad 2 \leq \alpha < \frac{1}{5}(q + r + 3). \tag{3.4}
\]

Concerning the viscosity, we assume that it does not depend on temperature and that \( s \mapsto \mu(s) \) satisfies for any \( s > 0 \)

\[
0 < \mu_0 \leq \mu(s) \leq \mu_1, \quad |\mu'(s)| \leq \mu_2 \tag{3.5}
\]

and/or

\[
|\mu''(s)| \leq \mu_3 \tag{3.6}
\]

for some positive constants \( \mu_0, \mu_1, \mu_2 \) and \( \mu_3 \).
3.2. The main results

**Theorem 3.1** Suppose that \((\eta_0, v_0, \theta_0, I_0) \in \mathcal{H}_2\) and the compatibility conditions hold. Then there exists a unique global solution \((\eta(t), v(t), \theta(t), I(t)) \in L^\infty([0, +\infty), \mathcal{H}_2)\) to the problem (1.1) satisfying for any \(t > 0\)

\[
\|\eta(t) - \bar{\eta}\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t) - \bar{\theta}\|_{H^2}^2 + \|I(t)\|_{H^1(\Omega \times R^+ \times S^1)}^2 + \|v_t(t)\|^2 + \|\theta_t(t)\|^2 \\
+ \int_0^t \left( \|v_{xt}\|^2 + \|\theta_{xt}\|^2 + \|\theta - \bar{\theta}\|_{H^3}^2 + \|v\|_{H^3}^2 + \|\eta - \bar{\eta}\|_{H^2}^2 \right)(s) ds \leq C_2.
\]

Moreover, as \(t \to +\infty\), we have

\[
\|\eta(t) - \bar{\eta}\|_{H^2} \to 0, \quad \|v(t)\|_{H^2} \to 0, \quad \|\theta(t) - \bar{\theta}\|_{H^2} \to 0, \quad \|I(t)\|_{H^1(\Omega \times R^+ \times S^1)} \to 0
\]

where \(\bar{\eta} = \int_0^1 \eta(x, t) dx = \int_0^1 \eta_0 dx, \bar{\theta} > 0\) is determined by \(e(\bar{\eta}, \bar{\theta}) = \int_0^1 \left( \frac{1}{2} v_0^2 + e(\eta_0, \theta_0) \right) dx\).
Theorem 3.2 Suppose that $(\eta_0, v_0, \theta_0, I_0) \in \mathcal{H}_3$ and the compatibility conditions hold. Then there exists a unique global solution $(\eta(t), v(t), \theta(t), I(t)) \in L^\infty([0, +\infty), \mathcal{H}_3)$ to the problem (1.1) verifying that for any $t > 0$,

$$
\|\eta(t) - \eta\|_{H^4}^2 + \|\eta_t(t)\|_{H^3}^2 + \|\eta_{tt}(t)\|_{H^1}^2 + \|v(t)\|_{H^4}^2 + \|v_{tt}(t)\|^2 + \|\theta(t) - \bar{\theta}\|_{H^4}^2 \\
+ \|\theta_t(t)\|_{H^2}^2 + \|\theta_{tt}(t)\|^2 + \|I\|_{H^3(\Omega \times R^+ \times S^1)}^2 + \int_0^t (\|\eta - \eta\|_{H^4}^2 + \|v\|_{H^5}^2 + \|v_t\|_{H^3}^2 \\
+ \|\theta - \bar{\theta}\|_{H^5}^2 + \|\theta_t\|_{H^3}^3 + \|\theta_{tt}\|_{H^1}^2)ds \leq C_4,
$$

$$
\int_0^t (\|\eta_t\|_{H^4}^2 + \|\eta_{tt}\|_{H^2}^2 + \|\eta_{ttt}\|^2)ds \leq C_4.
$$

Moreover, as $t \to +\infty$, we have

$$
\|\eta(t) - \eta\|_{H^4} \to 0, \quad \|v(t)\|_{H^4} \to 0, \quad \|\theta(t) - \bar{\theta}\|_{H^4} \to 0, \quad \|I(t)\|_{H^3(\Omega \times R^+ \times S^1)} \to 0.
$$
Corollary 3.1 The global solution \((\eta(t), v(t), \theta(t), I(t))\) obtained in Theorem 3.2 is in fact a classical solution and as \(t \to +\infty\), we have

\[
\|(\eta(t) - \bar{\eta}, v(t), \theta(t) - \bar{\theta})\|_{(C^{3+1/2})^3} \rightarrow 0, \quad \|I(t)\|_{C^{2+1/2}((0,1) \times \mathbb{R}^+ \times S^1)} \rightarrow 0.
\]

Remark 3.2 Theorems 3.1–3.2 also hold for the boundary conditions (1.2) and

\[
v(0, t) = v(1, t) = 0, \quad \theta(0, t) = \theta(1, t) = T_0 = \text{const.} > 0,
\]

where \(\bar{\theta}\) can be replaced by \(T_0 > 0\).
Global existence of solutions

In this section, we shall establish the global existence of solutions divided into several lemmas.

**Lemma 4.1** Assume that \( e, p, \sigma \) and \( \kappa \) are \( C^2 \) functions satisfying (3.1) on \( 0 < \eta < +\infty \) and \( 0 \leq \theta < +\infty \), and \( q, r, \alpha \) satisfy (3.4). Suppose that \((\eta_0, v_0, \theta_0, I_0) \in H_1\). Then the following estimates hold for any \( t > 0 \),

\[
\theta(x, t) > 0, \quad \forall (x, t) \in [0, 1] \times [0, +\infty), \tag{4.1}
\]

\[
\int_0^1 \eta(x, t)dx = \int_0^1 \eta_0(x)dx \equiv \eta_0, \quad \forall t > 0, \tag{4.2}
\]

\[
\int_0^1 (\theta + \theta^{1+r})(x, t)dx \leq C_1, \quad \forall t > 0, \tag{4.3}
\]

\[
\int_0^1 [((\theta - \log \theta - 1) + \theta^{1+r} + v^2](x, t)dx
\]

\[+ \int_0^t \int_0^1 \left( \frac{(1 + \theta^q)\theta^2_x}{\eta \theta^2} + \frac{\mu v^2_x}{\eta \theta} \right)(x, s)dxds \leq C_1, \tag{4.4}
\]

\[
\max_{[0, t]} \int_0^1 \int_0^{\infty} \int_{S^1} \eta I^2(x, t; \nu, \omega)d\omega d\nu dx \leq C_1, \tag{4.5}
\]

\[
0 < C_1^{-1} \leq \eta(x, t) \leq C_1. \tag{4.6}
\]

**Proof.** See, e.g., [8].
Lemma 4.2 Assume that $e$, $p$, $\sigma$ and $\kappa$ are $C^2$ functions satisfying (3.1) on $0 < \eta < +\infty$ and $0 \leq \theta < +\infty$, and $q$, $r$, $\alpha$ satisfy (3.4). Suppose that $(\eta_0, v_0, \theta_0, I_0) \in \mathcal{H}_1$. Then the following estimates hold for any $t > 0$,

\[
\int_0^t \int_0^1 (S_F)^2_R dx ds \leq C_1 (1 + \max_{(x,s) \in Q_t} \theta(x, s))^\alpha,
\]

\[
\int_0^t \int_0^1 (1 + \theta)^{q+r+1} v^2 dx ds \leq C_1,
\]

\[
\int_0^1 v^2 dx + \int_0^1 \eta_x^2 dx + \int_0^t \int_0^1 v_x^2 dx ds + \int_0^t \int_0^1 (1 + \theta^{r+1}) \eta_x^2 dx ds \\
\leq C_1 (1 + \max_{(x,s) \in Q_t} \theta(x, s))^\alpha.
\]
Lemma 4.3 Assume that $e$, $p$, $\sigma$ and $\kappa$ are $C^2$ functions satisfying (3.1) on $0 < \eta < +\infty$ and $0 \leq \theta < +\infty$, and $q$, $r$, $\alpha$ satisfy (3.4). Suppose that $(\eta_0, v_0, \theta_0, I_0) \in \mathcal{H}_1$. Then the following estimates hold for any $t > 0$,

$$\|v_x(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2 ds \leq C_1 (1 + \max_{(x,s) \in Q_t} \theta(x, s))^\beta,$$

(4.10)

$$\|\theta + \theta^{r+1}\|^2 + \int_0^t \int_0^1 (1 + \theta)^{q+r} \theta_x^2 dx ds \leq C_1 (1 + \max_{(x,s) \in Q_t} \theta(x, s))^{\beta_1},$$

(4.11)

where

$$\beta = \max \left\{ 3\alpha, r + 1 + \alpha \right\}, \quad \beta_1 = \max \left\{ \frac{r + 1 + \alpha}{2}, \gamma, q + 1 + \alpha, \frac{\alpha + \beta}{2} \right\},$$

$$\gamma = \min \left\{ \frac{(3r + 3 - 2q)_+}{2}, (2r + 1 - 2q)_+ \right\}, \quad A_+ = \max\{A, 0\}.$$
**Lemma 4.4** Assume that $e$, $p$, $\sigma$ and $\kappa$ are $C^2$ functions satisfying (3.1) on $0 < \eta < +\infty$ and $0 \leq \theta < +\infty$, and $q$, $r$, $\alpha$ satisfy (3.4). Suppose that $(\eta_0, v_0, \theta_0, I_0) \in H_1$. Then the following estimates hold for any $t > 0$,

\begin{align*}
\int_0^1 (1 + \theta^{2q})\theta_x^2 \, dx + \int_0^t \int_0^1 (1 + \theta)^{q+r} \theta_t^2 \, dx \, ds &\leq C_1, \quad (4.12) \\
\max_{(x,s) \in Q_t} \theta(x,s) &\leq C_1, \quad (4.13) \\
\|\eta_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_x(t)\|^2 + \|\theta + \theta^{1+r}\|_2^2 + \int_0^t (\|v_x\|^2 + \|v_{xx}\|^2 + \|\theta_x\|^2 + \|\theta_{xx}\|^2 + \|\theta_t\|^2 + \|v_t\|^2) \, ds &\leq C_1. \quad (4.14)
\end{align*}
**Proof.** Let

\[ K(\eta, \theta) = \int_0^\theta \frac{\kappa(\eta, u)}{\eta} du, \]

\[ X(t) = \int_0^t \int_0^1 (1 + \theta^{q+r})\theta^2_t dx ds, \quad Y(t) = \int_0^1 (1 + \theta^2)\theta^2_x dx. \]

Then it is easy to verify that

\[ K_t = K_\eta v_x + \frac{\kappa}{\eta} \theta_t, \quad K_{xt} = \left( \frac{\kappa \theta_t}{\eta} \right)_t + K_{\eta_1} v_x \eta_x + \left( \frac{\kappa}{\eta} \right) \eta_x \theta_t + K_\eta v_{xx}. \]

We know that

\[ |K_\eta| + |K_{\eta_1}| \leq C_1(1 + \theta^{q+1}). \]

The third equation in (1.1) can be rewritten as

\[ e_\theta \theta_t + \theta p_\theta v_x - \frac{\mu}{\eta} v_x^2 = \left( \frac{\kappa \theta_x}{\eta} \right)_x. \]

Multiplying the above inequality by \( K_t \) and integrating the result over \( Q_t \), we arrive at

\[ \int_0^t \int_0^1 \left( e_\theta \theta_t + \theta p_\theta v_x - \frac{\mu}{\eta} v_x^2 \right) K_t dx ds + \int_0^t \int_0^1 \left( \frac{\kappa \theta_x}{\eta} \right) K_{tx} dx ds = 0. \]
We have

\[
\int_0^t \int_0^1 \kappa e_\theta \theta_t^2 dx ds \geq C_1^{-1} \int_0^t \int_0^1 (1 + \theta)^q r \theta_t^2 dx ds = C_1^{-1} X(t).
\]

We deduce for any \(\varepsilon > 0\)

\[
\left| \int_0^t \int_0^1 e_\theta \theta_t K_\eta v_x dx ds \right| \leq C_1 \int_0^t \int_0^1 (1 + \theta)^q r \theta_t^2 |\theta_t v_x| dx ds
\]

\[
\leq \varepsilon \int_0^t \int_0^1 (1 + \theta)^q r \theta_t^2 dx ds + C_\varepsilon \int_0^t \int_0^1 (1 + \theta)^q r \theta_t^2 v_x^2 dx ds
\]

\[
\leq \varepsilon X(t) + C_1 (1 + \max_{s \in [0,t]} \theta(x, s))^{q+r+2+\alpha},
\]

and

\[
\int_0^t \int_0^1 \left( \theta p \theta v_x - \frac{\mu}{\eta} v_x^2 \right) \left( K_\eta v_x + \frac{\kappa}{\eta} \theta_t \right) dx ds \leq \varepsilon X(t) + C_1 (1 + \max_{s \in [0,t]} \theta(x, s))^{\beta_2},
\]

where

\[
\beta_2 = \max \left\{ q + r + 2 + \alpha, q + \frac{\alpha + 3\beta}{2} + 1 \right\}.
\]
Similarly, we can derive
\[
\left| \int_0^t \int_0^1 \left( \frac{\kappa \theta_x}{\eta} \right)_t \left( \frac{\kappa \theta_x}{\eta} \right) \, dx \, ds \right| \geq C_1^{-1} Y(t) - C_1,
\]
\[
\left| \int_0^t \int_0^1 \frac{\kappa \theta_x}{\eta} (K_\eta v_{xx} + K_\eta v_x \eta_x) \, dx \, ds \right| \leq C_1 (1 + \max_{s \in [0, t]} \theta(x, s))^{\beta_4},
\]
where
\[
\beta_4 = \max \left\{ \frac{3q + 4 + \beta}{2}, \frac{3q + 4}{2} + \frac{3\alpha + \beta}{4} \right\}.
\]
We derive for any \( \varepsilon > 0 \),
\[
\left| \int_0^t \int_0^1 \frac{\kappa \theta_x}{\eta} \left( \frac{\kappa}{\eta} \right)_t \eta_x \theta_t \, dx \, ds \right| \leq \varepsilon X(t) + C_1 (1 + \max_{s \in [0, t]} \theta(x, s))^{\beta_6},
\]
where

\[ \beta_6 = \max\{2q + 2 + \alpha - r, \beta_5\}, \]

and

\[ \beta_5 = \max\left\{ \frac{3q}{2} + 2 + 2\alpha, (2q + 2\alpha + 1 - r)_+, \frac{3q + 3\alpha + \beta + 2}{2} - r, \frac{3q + 2 - 2r}{2} + \frac{5\alpha + \beta}{4} \right\}. \]

Then, we arrive at

\[ X(t) + Y(t) \leq C_1(1 + \max_{s \in [0,t]} \theta(x, s))^{\beta_7}, \]

where

\[ \beta_7 = \max\{q + r + 2 + \alpha, \beta_2, \beta_6, \beta_4\}. \]
The embedding theorem, previous lemmas and Young’s inequality result in

\[
\left\| \theta(t) - \int_0^1 \theta \, dx \right\|_{L^\infty} \leq \int_0^1 |\theta^{q+(r+1)/2} \theta_x| \, dx \\
\leq C_1 Y^{1/2}(t) \left( \int_0^1 \theta^{r+1} \, dx \right)^{1/2} \leq C_1 Y^{1/2}(t),
\]

which gives

\[
\left\| \theta(t) \right\|_{L^\infty}^{2q+r+3} \leq C_1 + C_1 Y(t).
\]

Noting that (3.4) implies that \( \beta_7 < 2q + r + 3 \), hence we get for any \( \varepsilon > 0 \),

\[
X(t) + Y(t) \leq C_1 + C_1 Y^{\beta_7/(2q+r+3)}(t) \\
\leq \varepsilon Y(t) + C_1.
\]

Taking \( \varepsilon > 0 \) small enough, we get

\[
X(t) + Y(t) \leq C_1,
\]
In [8], the authors have obtained the following lemma

**Lemma 4.5** Assume that \( e, p, \sigma \) and \( \kappa \) are \( C^2 \) functions satisfying (3.1) on \( 0 < \eta < +\infty \) and \( 0 \leq \theta < +\infty \), and \( q, r, \alpha \) satisfy (3.4). Suppose that \( (\eta_0, v_0, \theta_0, I_0) \in \mathcal{H}_1 \). Then the following estimates hold for any \( t > 0 \),

\[
\max_{s \in [0,t]} \int_0^1 \int_0^\infty \int_{S^1} I_t^2(x, s; \nu, \omega) d\omega d\nu dx \leq C_1, \tag{4.15}
\]

\[
\lim_{t \to +\infty} (||\eta - \bar{\eta}||_{H^1} + ||v||_{H^1} + ||\theta - \bar{\theta}||_{H^1} + ||I||_{L^2(\Omega \times R^+ \times S^1)}) = 0. \tag{4.16}
\]

The next lemma concerns the uniform-in-time global (in time) positive lower bound (independent of \( t \)) of the absolute temperature \( \theta \).

**Lemma 4.6** Under the assumptions in Theorem 3.1, then the generalized global solution to the problem satisfies

\[
0 < C_1^{-1} \leq \theta(x, t), \quad \forall (x, t) \in [0, 1] \times [0, +\infty). \tag{4.17}
\]

**Proof.** See, e.g., Lemma 2.3.3 on page 85 of [22].
Lemma 4.7 Under the assumptions in Theorem 3.1, the following estimates hold

\[ \|v_t(t)\|^2 + \int_0^t \|v_{xt}(s)\|ds \leq C_2, \]  
(4.18)

\[ \|v_{xx}(t)\|^2 \leq C_2, \]  
(4.19)

\[ \|\theta_{xx}(t)\|^2 + \int_0^t (\|\theta_{xxx}\|^2 + \|\theta_{xt}\|^2)(s)ds \leq C_2. \]  
(4.20)

Lemma 4.8 Under the assumptions in Theorem 3.1, the following estimate holds

\[ \|\eta_{xx}(t)\|^2 + \int_0^t \|\eta_{xx}(s)\|^2 ds \leq C_2 + C_2 \int_0^t \int_0^1 \int_\infty \int_{S^1} \sigma_s(\tilde{I} - I)^2 d\omega d\nu dx ds, \]  
(4.21)

where \( \tilde{I} = \frac{1}{2} \int_{S^1} I(x, t; \nu, \omega)d\omega. \)
Lemma 4.9 Under the assumptions in Theorem 3.1, the following estimate holds

\[
\max_{[0,t]} \int_0^1 \int_0^\infty \int_{S^1} I_2^2 d\omega d\nu dx \leq C_2.
\]  

(4.22)

Proof. Going back to Eulerian coordinates, \( I(y, \tau; \nu, \omega) \) solves the problem

\[
\begin{cases}
I_\tau(y, \tau; \nu, \omega) + c\omega I_y(y, \tau; \nu, \omega) = c\sigma_s(\nu; \eta, \theta)[\tilde{I} - I] := S(I; y, \tau; \nu, \omega), \\
I(0, \tau; \nu, \omega) = 0 \text{ for } \omega \in (0, 1), \\
I(L, \tau; \nu, \omega) = 0 \text{ for } \omega \in (-1, 0), \\
I(y, 0; \nu, \omega) = I_0(y; \nu, \omega) \text{ on } \mathcal{O} \times R^+ \times S^1.
\end{cases}
\]  

(4.23)

where \( \mathcal{O} = (0, L) \) with \( L > 0 \) is the domain in Eulerian coordinates.

Differentiating the equation with respect to \( y \) and denoting \( G = I_y \), we can derive that \( G \) solves the problem

\[
\begin{cases}
G_\tau + c\omega G_y = S_y, \\
G(0, \tau; \nu, \omega) = 0 \text{ for } \omega \in (0, 1), \\
G(L, \tau; \nu, \omega) = 0 \text{ for } \omega \in (-1, 0), \\
G(y, 0; \nu, \omega) = G_0(y; \nu, \omega) \text{ on } \mathcal{O} \times R^+ \times S^1,
\end{cases}
\]  

(4.24)

where \( S_y = S(G; y, \tau; \nu, \omega) + \phi(G; y, \tau; \nu, \omega) \).
Multiplying (4.24) by $G$, using the previous lemmas and Young’s inequality, we get

$$\frac{1}{2} \int_0^\infty \int_{S^1} G^2 d\omega d\nu dy - \frac{1}{2} \int_0^\infty \int_{S^1} G_0^2 d\omega d\nu dy + c \int_0^\infty \int_{S^1} \omega G^2(L, \tau; \nu, \omega) d\omega d\nu$$

$$- c \int_0^\infty \int_{S^1} \omega G^2(0, \tau; \nu, \omega) d\omega d\nu + \int_0^\tau \int_0^\infty \int_{S^1} \eta \sigma_s (\tilde{G} - G)^2 d\omega d\nu dy dt$$

$$\leq \frac{1}{2} \int_0^\tau \int_0^\infty \int_{S^1} \phi^2 d\omega d\nu dy dt. \quad (4.25)$$

Using the coordinates transformation and precious lemmas, we have

$$\int_0^1 \int_0^\infty \int_{S^1} I_x^2 d\omega d\nu dx \leq C_2 + C_2 \left| \int_0^t \int_0^1 \int_{S^1} \sigma_s (\tilde{I} - I)_x d\omega d\nu dx ds \right|$$

$$\leq C_2 + C_2 \int_0^t \|\theta x\|^2 \int_0^1 \int_{S^1} I_x^2 d\omega d\nu dx ds.$$ 

Applying Gronwall’s inequality, we complete the proof.
**Lemma 4.10** Under the assumptions in Theorem 3.1, the following estimates hold

\[ \|\eta_{xx}(t)\|^2 + \int_0^t \|\eta_{xx}(s)\|^2 ds \leq C_2, \]  
(4.26)

\[ \int_0^t \|v_{xxx}(s)\|^2 ds \leq C_2. \]  
(4.27)

Now we have proved the existence of solutions in $H_2$. 

Lemma 4.11 Under the assumptions in Theorem 3.2, the following estimates hold

\[ ||v_{xt}(x, 0)|| + ||\theta_{xt}(x, 0)|| \leq C_3, \tag{4.28} \]
\[ ||v_{tt}(x, 0)|| + ||\theta_{tt}(x, 0)|| + ||v_{xxt}(x, 0)|| + ||\theta_{xxt}(x, 0)|| \leq C_4, \tag{4.29} \]
\[ ||v_{tt}(t)||^2 + \int_0^t ||v_{xtt}(s)||^2 ds \leq C_4 + C_2 \int_0^t (||\theta_{tt}(s)||^2_{H^1} + ||v_x(s)||^2_{H^3}) \]
\[ + ||\theta_x(s)||^2 ||I_x(s)||^2_{H^1(\Omega \times R^+ \times S^1)} ds, \tag{4.30} \]
\[ ||\theta_{tt}(t)||^2 + \int_0^t ||\theta_{xtt}(s)||^2 ds \leq C_4 \varepsilon^{-3} + C_2 \varepsilon^{-1} \int_0^t ||\theta_{xxt}(s)||^2 ds \]
\[ + C_1 \varepsilon \int_0^t (||v_{xtt}||^2 + ||v_{xxt}||^2)(s) ds. \tag{4.31} \]
Lemma 4.12 Under the assumptions in Theorem 3.2, the following estimates hold

\[ \|\nu_{xt}(t)\|^2 + \int_0^t \|\nu_{xxt}(s)\|^2 ds \leq C_3 \varepsilon^{-6} + C_1 \varepsilon^2 \int_0^t (\|\theta_{xxt}\|^2 + \|\nu_{xtt}\|^2)(s) ds \]

\[ + C_3 \varepsilon^{-6} \int_0^t \|\theta_{x}(s)\|^2 \|I_x(s)\|^2_{H^1(\Omega \times R^+ \times S^1)} ds, \quad (4.32) \]

\[ \|\theta_{xt}(t)\|^2 + \int_0^t \|\theta_{xxt}(s)\|^2 ds \leq C_3 \varepsilon^{-6} + C_1 \varepsilon^2 \int_0^t (\|\nu_{xxt}\|^2 + \|\theta_{xtt}\|^2)

\[ + \|\theta_{xxx}\|^2 \|\theta_{xt}\|^2)(s) ds. \quad (4.33) \]
Similarly to Lemma 4.9, we can get

**Lemma 4.13** Under the assumptions in Theorem 3.2, the following estimate holds for any $\varepsilon > 0$,

$$\max_{s \in [0,t]} \int_0^1 \int_0^\infty \int_{S^1} I_{xx}^2(x, s; \nu, \omega) d\omega d\nu dx \leq C_2 + \varepsilon \int_0^t (\|\eta'(s)\|_{H^2}^2 + \|\theta'(s)\|_{H^2}^2) ds. \quad (4.34)$$
Lemma 4.14 Under the assumptions in Theorem 3.2, the following estimates hold

\[
\|v_{tt}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \int_0^t (\|v_{xtt}\|^2 + \|v_{xtt}\|^2 \\
+ \|\theta_{xtt}\|^2 + \|\theta_{xxt}\|^2(s)ds \leq C_4,
\]  
(4.35)

\[
\max_{s\in[0,t]} \int_0^1 \int_0^\infty \int_{S^1} I_{xxt}^2(x,s;\nu,\omega)d\omega d\nu dx \leq C_4,
\]  
(4.36)

\[
\|\eta_{xxx}(t)\|_{H^1}^2 + \|\eta_{xx}(t)\|_{W^{1,\infty}}^2 + \int_0^t (\|\eta_{xxx}\|_{H^1}^2 + \|\eta_{xx}\|_{W^{1,\infty}}^2)(s)ds \leq C_4,
\]  
(4.37)

\[
\|v_{xxx}(t)\|_{H^1}^2 + \|v_{xx}(t)\|_{W^{1,\infty}}^2 + \|\theta_{xxx}(t)\|^2 + \|\theta_{xx}\|_{W^{1,\infty}}^2 + \|\eta_{xxx}(t)\|^2 \\
+ \|v_{xxt}(t)\|^2 + \|\theta_{xxt}\|^2 + \int_0^t (\|v_{tt}\|^2 + \|\theta_{tt}\|^2 + \|v_{xx}\|_{W^{2,\infty}}^2 + \|\theta_{xx}\|_{W^{2,\infty}}^2 \\
+ \|\theta_{xxt}\|_{H^1}^2 + \|\theta_{xxt}\|_{H^1}^2) + \|v_{xxt}\|_{W^{1,\infty}}^2 + \|v_{xt}\|_{W^{1,\infty}}^2 + \|\eta_{xxx}\|_{H^1}^2)(s)ds \leq C_4,
\]  
(4.38)

\[
\int_0^t (\|v_{xxxx}\|_{H^1}^2 + \|\theta_{xxxx}\|_{H^1}^2)(s)ds \leq C_4.
\]  
(4.39)

We proved the global existence of solutions.
5 Large-time behavior of solutions

In this section, we will prove the large-time behavior of solutions. In the following lemma, we introduce a differential inequality, which will play an essential role in the proof of large-time behavior of solutions.

**Lemma 5.1** Let $T$ be given with $0 < T \leq +\infty$. Suppose that $y$ and $h$ are nonnegative continuous functions defined on $[0, T]$ and satisfy the following conditions

\[
\frac{dy}{dt} \leq A_1 y^2(t) + A_2 + h(t),
\]

\[
\int_0^T y(s)ds \leq A_3, \quad \int_0^T h(s)ds \leq A_4,
\]

where $A_1, A_2, A_3, A_4$ are given nonnegative constants. Then for any $r > 0$, with $0 < r < T$,

\[
y(t + r) \leq \left( \frac{A_3}{r} + A_2r + A_4 \right) \cdot e^{A_1A_3}.
\]

Furthermore, if $T = +\infty$, then

\[
\lim_{t \to +\infty} y(t) = 0.
\]

**Proof.** See, e.g., [26].
Lemma 5.2 Under the assumptions in Theorem 3.1, the following estimates hold

\[
\lim_{t \to +\infty} ||\eta(t) - \bar{\eta}||_{H^2} = 0, \tag{5.1}
\]
\[
\lim_{t \to +\infty} ||v(t)||_{H^2} = 0, \tag{5.2}
\]

where \( \bar{\eta} = \int_0^1 \eta(y, t)dy = \int_0^1 \eta_0(y)dy. \)

Proof. Using previous lemmas, we derive

\[
\frac{d}{dt} ||\eta_{xx}(t)||^2 \leq ||\eta_{xx}(t)||^2 + ||v_{xx}(t)||^2, \tag{5.3}
\]

which gives

\[
||\eta_{xx}(t)|| \to 0, \quad as \ t \to +\infty. \tag{5.4}
\]

Similarly, we have

\[
\frac{d}{dt} ||v_t(t)||^2 + C_2^{-1}||v_{xt}(t)||^2 \leq C_2(||v_x(t)||_{H^1}^2 + ||\theta(t)||^2 + ||I_t(t)||_{L^2(\Omega \times R^+ \times S^1)}).
\]

Then we get

\[
||v_t(t)|| \to 0, \quad as \ t \to +\infty. \tag{5.5}
\]
Similarly, we obtain

\[ \frac{d}{dt} \|(S_F)_R\|^2 \leq C_2 + C_2(\|v_x\|_{H^1}^2 + \|\theta_t(t)\|_{H^1}^2), \]

which yields

\[ \|(S_F)_R\| \to 0, \quad as \ t \to +\infty. \quad (5.6) \]

Noting that

\[ \|v_{xx}(t)\| \leq C_2(\|v_t(t)\| + \|\eta_x(t)\|_{H^1} + \|v_x(t)\| + \|\theta_x(t)\| + \|(S_F)_R\|), \]

we get

\[ \|v_{xx}(t)\| \to 0, \quad as \ t \to +\infty. \quad (5.7) \]
Lemma 5.3 Under the assumptions in Theorem 3.1, we have

$$\lim_{t \to +\infty} ||\theta(t) - \bar{\theta}||_{H^2} = 0,$$

(5.8)

where \( \bar{\theta} > 0 \) is determined by \( e(\bar{\eta}, \bar{\theta}) = \int_0^1 \left( \frac{1}{2} v_0^2 + e(\eta_0, \theta_0) \right) dx \).

Proof. We deduce

$$\frac{d}{dt} ||\theta_t(t)||^2 + ||\theta_{xt}(t)||^2 \leq C_2 (||\theta_x(t)||^2 + ||v_x(t)||^2 + ||\theta_{xt}(t)||^2 + ||\theta_t(t)||^2 + ||v_{xt}(t)||^2),$$

(5.9)

which gives

$$||\theta_t(t)|| \to 0, \quad \text{as } t \to +\infty.$$

(5.10)

Noting that

$$||\theta_{xx}(t)|| \leq C_2 (||v_x(t)||_{H^1} + ||\eta_x(t)||_{H^1} + ||\theta_t(t)|| + ||\theta_x(t)|| + ||(S_F)_{R}||),$$

then we get

$$||\theta_{xx}(t)|| \to 0, \quad \text{as } t \to +\infty.$$
Lemma 5.4 Under the assumptions in Theorem 3.1, we have

$$\lim_{t \to +\infty} \left\| I(t) \right\|_{H^1(\Omega \times R^+ \times S^1)} = 0.$$  \hfill (5.12)

Proof. From the last equation of (1.1), we derive

$$\frac{d}{dt} \left\| I_x(t) \right\|_{L^2(\Omega \times R^+ \times S^1)}^2 \leq C_2 \left( \left\| \eta - \eta \right\|^2_{H^2} + \left\| v \right\|^2_{H^2} + \left\| \theta - \theta \right\|^2_{H^2} \right) + C_2,$$

which yields

$$\left\| I_x(t) \right\|_{L^2(\Omega \times R^+ \times S^1)} \to 0, \text{ as } t \to +\infty.$$ \hfill (5.13)

Now we complete the proof of Theorem 3.1.
The same proof as large-time behavior of solutions in $\mathcal{H}_2$, we can prove the large-time behavior of solutions in $\mathcal{H}_3$.

**Lemma 5.5** *Under the assumptions in Theorem 3.2, we have*

\[
\begin{align*}
    \lim_{t \to +\infty} ||\eta(t) - \eta||_{H^4} &= 0, \quad (5.14) \\
    \lim_{t \to +\infty} ||v(t)||_{H^4} &= 0, \quad (5.15) \\
    \lim_{t \to +\infty} ||\theta(t) - \bar{\theta}||_{H^4} &= 0, \quad (5.16) \\
    \lim_{t \to +\infty} ||I(t)||_{H^3(\Omega \times R^+ \times S^1)} &= 0 \quad (5.17)
\end{align*}
\]

*where $\eta = \int_0^1 \eta(y, t)dy$, $\bar{\theta} > 0$ is determined by $e(\eta, \bar{\theta}) = \int_0^1 (\frac{1}{2}v_0^2 + e(\eta_0, \theta_0))dx$.**
Proof. We can derive

\[
\frac{d}{dt} \left\| \frac{\mu \eta_{xxx}}{\eta} \right\| + C_1^{-1} \left\| \frac{\eta_{xxx}}{\eta} \right\| \leq C_2 (\| \eta_x(t) \|_{H^1} + \| v_x(t) \|_{H^2} + \| \theta_x(t) \|_{H^2} + \| v_{xx}(t) \| \\
+ \| \theta_x(t) \| \| I_{xx}(t) \|_{L^2((0,1) \times R^+ \times S^1)}) ,
\]

which gives

\[
\lim_{t \to +\infty} \| \eta_{xxx}(t) \|^2 = 0.
\] (5.19)

Also,

\[
\frac{d}{dt} \left\| \frac{\mu \eta_{xxxx}}{\eta} \right\| + C_1^{-1} \left\| \frac{\eta_{xxxx}}{\eta} \right\| \leq C_1 \| v_{xxxx}(t) \| + C_4 (\| \theta_x(t) \|_{H^3} + \| \eta_x(t) \|_{H^2} + \| v_x(t) \|_{H^3} \\
+ \| \theta_x(t) \| \| I_{xxx}(t) \|_{L^2((0,1) \times R^+ \times S^1)}) ,
\]

which gives

\[
\lim_{t \to +\infty} \| \eta_{xxxx}(t) \|^2 = 0.
\] (5.21)
Firstly,

\[
\frac{d}{dt} \|v_{xt}(t)\|^2 + C_2^{-1} \|v_{xxx}(t)\|^2 \leq \varepsilon^2 (\|v_{xxx}(t)\|^2 + \|\theta_{xt}(t)\|^2) + C_\varepsilon (\|v_{x}(t)\|_{H^2}^2 + \|\theta_{x}(t)\|_{H^1}^2)
\]

\[
+ \|\theta_{xt}(t)\|_{H^1}^2 + \|v_{xt}\|^2 + \|\theta_x(t)\|_{H^1}^2
\]

\[
+ \|\theta_x(t)\| \|I_x(t)\|_{H^1((0,1) \times R^+ \times S^1)}
\]

\[
+ \|\theta_x(t)\| \|I_t(t)\|_{L^2((0,1) \times R^+ \times S^1)}
\]

which yields

\[
\|v_{xt}(t)\|^2 \rightarrow +\infty \text{ as } t \rightarrow +\infty.
\]

which, together with

\[
\|v_{xxx}(t)\| \leq C_2 (\|v(t)\|_{H^2} + \|\eta_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^1} + \|v_{xt}(t)\|)
\]

\[
+ \|\theta_x(t)\| \|I_x(t)\|_{L^2((0,1) \times R^+ \times S^1)}
\]

gives

\[
\lim_{t \to +\infty} \|v_{xxx}(t)\| = 0.
\]
From the second equation in (1.1), we derive
\[
\frac{d}{dt} \left| v_{xx}(t) \right|^2 + C_4^{-1} \left| v_{xxx}(t) \right|^2 \leq C_1 \left| \theta_x(t) \right|^2_{H^2} + C_2 \left( \left| v_{x}(t) \right|^2_{H^2} + \left| v_{x}(t) \right|^2_{H^2} + \left| \eta_x(t) \right|^2_{H^2} + \left| \theta_{xx}(t) \right|^2_{H^2} + \left| \theta_{xx}(t) \right|^2_{H^2} + \left| \theta_{x}(t) \right|^2_{H^2} + \left| \theta_{x}(t) \right|^2_{H^2} + \left| I_x(t) \right|^2_{H^1((0,1) \times R^+ \times S^1)} \right)
\]

This implies
\[
\lim_{t \to +\infty} \left| v_{xx}(t) \right|^2 = 0. \tag{5.26}
\]

Similarly, we get
\[
\lim_{t \to +\infty} \left| \theta_{xx}(t) \right|^2 = 0, \tag{5.27}
\]

which, combined with
\[
\left| v_{xxxx}(t) \right| \leq C_2 \left( \left| \eta_x(t) \right|^2_{H^2} + \left| v_x(t) \right|^2_{H^2} + \left| \theta_x(t) \right|^2_{H^2} + \left| v_{xx}(t) \right|^2_{H^2} + \left| \theta_{xx}(t) \right|^2_{H^2} + \left| \theta_{x}(t) \right|^2_{H^2} + \left| I_x(t) \right|^2_{H^1((0,1) \times R^+ \times S^1)} \right), \tag{5.28}
\]

implies
\[
\lim_{t \to +\infty} \left| v_{xxxx}(t) \right| = 0. \tag{5.29}
\]
By the third equation in (1.1), we deduce

\[
\lim_{t \to +\infty} \|\theta_{tt}(t)\|^2 = 0. \tag{5.30}
\]

Similarly, we get

\[
\lim_{t \to +\infty} \|\theta_{xxt}(t)\|^2 = 0, \tag{5.31}
\]

which, along with

\[
\|\theta_{xxxx}(t)\| \leq C_2 (\|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_{xxt}(t)\| + \|\theta_x(t)\| \|I_x(t)\|_{H^1((0,1) \times R^+ \times S^1)}), \tag{5.32}
\]

gives

\[
\lim_{t \to +\infty} \|\theta_{xxxx}(t)\| = 0. \tag{5.33}
\]
A direct computation gives
\[
\frac{d}{dt} \left\| I_{xx}(t) \right\|_{L^2((0,1) \times R^+ \times S^1)} \leq C_1 \left( \left\| \eta_x(t) \right\|_{H^2}^2 + \left\| \theta_x(t) \right\|_{H^2}^2 + \left\| v_x(t) \right\|_{H^2}^2 \\
+ \left\| I_t(t) \right\|_{L^2((0,1) \times R^+ \times S^1)}^2 + \left\| I_x(t) \right\|_{H^1((0,1) \times R^+ \times S^1)}^2 \right),
\]
and
\[
\frac{d}{dt} \left\| I_{xxx}(t) \right\|_{L^2((0,1) \times R^+ \times S^1)}^2 \leq C_1 \left( \left\| \eta_x(t) \right\|_{H^3}^2 + \left\| \theta_x(t) \right\|_{H^3}^2 + \left\| v_x(t) \right\|_{H^3}^2 \\
+ \left\| I_t(t) \right\|_{H^1((0,1) \times R^+ \times S^1)}^2 + \left\| I_x(t) \right\|_{H^2((0,1) \times R^+ \times S^1)}^2 \right),
\]
which implies
\[
\lim_{t \to +\infty} \left\| I_{xx}(t) \right\|_{L^2((0,1) \times R^+ \times S^1)}^2 = 0 \tag{5.34}
\]
and
\[
\lim_{t \to +\infty} \left\| I_{xxx}(t) \right\|_{L^2((0,1) \times R^+ \times S^1)}^2 = 0. \tag{5.35}
\]
6 The case of multi-dimensional model

If the matter is at local thermodynamics equilibrium, the system for the density \( \rho(x, t) \), velocity \( \vec{u}(x, t) \), temperature \( \theta(x, t) \) and radiative intensity \( I(x, t, \vec{\Omega}, \nu) \) reads

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0, \\
(\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) &= -\nabla \cdot \vec{\Pi} + \vec{S}_F, \\
(\rho \varepsilon)_t + \nabla \cdot (\rho \varepsilon \vec{u}) &= -\nabla \vec{q} - \vec{D} : \vec{\Pi} + S_E, \\
\frac{1}{c} \frac{\partial}{\partial t} I(r, t, \vec{\Omega}, \nu) + \vec{\Omega} \cdot \nabla I(r, t, \vec{\Omega}, \nu) &= S_t(r, t, \vec{\Omega}, \nu),
\end{aligned}
\]  

(6.1)

where \((x, t, \vec{\Omega}, \nu) \in R^3 \times [0, T] \times S^2 \times R, \vec{\Omega} \) and \( \nu \) are the angular variable and the frequency of the radiation, and where \( \vec{\Pi} = -p(\rho, T) \vec{I} + \vec{\pi} \) is the material stress tensor for a newtonian fluid with the viscous contribution \( \vec{\pi} = 2\mu \vec{D} + \lambda \nabla \cdot \vec{u} \vec{I} \) with \( 3\lambda + 2\mu \geq 0 \) and \( \mu > 0 \), and the strain tensor \( \vec{D} \) such that \( \vec{D}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \), the coupling terms are

\[
S_t(r, t, \vec{\Omega}, \nu) = \sigma_a \left( \nu, \vec{\Omega}, \rho, T, \frac{\vec{\Omega} \cdot \vec{u}}{c} \right) \left[ B(\nu, T) - I(r, t, \vec{\Omega}, \nu) \right]
\]

\[
+ \int \int \sigma_s(r, t, \rho, \vec{\Omega}' \cdot \vec{\Omega}, \nu' \rightarrow \nu) \times \left\{ \frac{\nu}{\nu'} I(r, t, \vec{\Omega}', \nu') I(r, t, \vec{\Omega}, \nu) - \sigma_s(r, t, \rho, \vec{\Omega}' \cdot \vec{\Omega}, \nu' \rightarrow \nu) I(r, t, \vec{\Omega}', \nu') I(r, t, \vec{\Omega}, \nu) \right\} d\Omega' d\nu'
\]
the radiative energy source

\[ S_E(r, t) := \int \int S_t(r, t, \vec{\Omega}, \nu) d\Omega d\nu, \]

the radiative flux

\[ \mathbf{S}_F(r, t) := \frac{1}{c} \int \int \mathbf{\Omega} S_t(r, t, \vec{\Omega}, \nu) d\Omega d\nu, \]

the functions \( \sigma_a \) and \( \sigma_s \) describe in a phenomenological way the absorption-emission and scattering properties of the photon-matter interaction, and Planck’s function \( B(\nu, \theta) \) describes the frequency-temperature black body distribution.
Remark 6.1 The multi-dimensional viscous situation has been poorly understood even at the formal level. Since the one-dimensional model possesses the special constitutive state equations which the multi-dimensional models don’t have, to our knowledge, we have not found any results on the global existence and asymptotic behavior of solutions to system (6.1), i.e., the multi-dimensional case of system (2.1). Moreover, some Sobolev embedding inequalities and interpolation inequalities involved in our arguments heavily depend on the dimension, hence this may bring about some difficulties in deriving uniform-in-time estimates. In a word, the method we deal with the one-dimensional case can not be applied directly to the multi-dimensional case, which depends on the special constitutive relations of state functions, and so on.
7 References


Thank you for your attention!