Global Existence and Asymptotic Behavior of Solutions for Thermodiffusion Equations

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Thermodiffusion Equations

We consider the initial boundary value problems for the 1-d thermodiffusion equations in a bounded region $\Omega = (0, 1)$. The thermodiffusion equations describe the process of thermodiffusion in a solid body. Such a process is described by the following system of equations (see, e.g., W. Nowacki, 1971, 1974):

\begin{align*}
\rho u_{tt} - (\lambda + 2\mu) u_{xx} + \gamma_1 \theta_1 x + \gamma_2 \theta_2 x &= f, \quad \text{in } \Omega \times \mathbb{R}^+, \\
c \theta_1 t - k \theta_1 xx + \gamma_1 u_{tx} + d \theta_2 t &= g, \quad \text{in } \Omega \times \mathbb{R}^+, \\
n \theta_2 t - D \theta_2 xx + \gamma_2 u_{tx} + d \theta_1 t &= h, \quad \text{in } \Omega \times \mathbb{R}^+, 
\end{align*}

where $\mathbb{R}^+ = (0, +\infty)$, $u(x, t)$, $\theta_1(x, t)$, and $\theta_2(x, t)$ are the displacement, temperature, and chemical potential. Functions $f$, $g$ and $h$ are known functions specified later on,
Thermodiffusion Equations

together with the initial conditions

\[ u(x,0)=u_0(x), \ u_t(x,0)=u_1(x), \ \theta_1(x,0)=\theta_{10}(x), \ \theta_2(x,0)=\theta_{20}(x), \]  

(1.4)

and one of the following boundary conditions:

\[ u(x,t)|_{x=0,1} = \theta_1(x,t)|_{x=0,1} = \theta_2(x,t)|_{x=0,1} = 0, \]  

(1.5)

\[ u(x,t)|_{x=0,1} = \theta_1(x,t)|_{x=0,1} = \theta_2(x,t)|_{x=0,1} = 0, \]  

(1.6)

\[ u(x,t)|_{x=0,1} = \theta_1(x,t)|_{x=0,1} = \theta_2(x,t)|_{x=0,1} = 0, \]  

(1.7)

\[ u(x,t)|_{x=0,1} = \theta_1(x,t)|_{x=0,1} = \theta_2(x,t)|_{x=0,1} = 0, \]  

(1.8)

\[ u_x(x,t)|_{x=0,1} = \theta_1(x,t)|_{x=0,1} = \theta_2(x,t)|_{x=0,1} = 0. \]  

(1.9)

Firstly, we are going to consider the equations (1.1)-(1.3) with the initial conditions (1.4) and the first boundary conditions (1.5). The problem (1.1)-(1.4) with one of the other boundary conditions (1.6)-(1.9) shall be considered at the end.
Thermodiffusion Equations

Here, we denote $\lambda$, $\mu$ the material coefficients, $\rho$ the density, $\gamma_1$, $\gamma_2$ the coefficients of thermal and diffusion dilatation, $k$ the coefficient of thermal conductivity, $D$ the coefficient of thermal conductivity, $n$, $c$, $d$ the coefficients of thermodiffusion. All the above coefficients are positive constants and satisfy

$$nc - d^2 > 0. \quad (1.10)$$

(1.10) implies that (1.1)-(1.3) is a hyperbolic-parabolic system and similar with the one-dimensional linear thermoelastic system.

The total energy for these problems is given by

$$E(t) = \int_0^1 [(\lambda + 2\mu)u_x^2 + \rho u_t^2 + c\theta_1^2 + n\theta_2^2 + 2d\theta_1\theta_2] dx. \quad (1.11)$$
We first recall some known results.


- Gawinecki (1982) proved the existence, uniqueness and regularity of solutions to the initial boundary value problem for the linear system of thermodiffusion in a solid body.

- Using the Fourier transform, the matrix of fundamental solutions was constructed by Gawinecki (1991) for three cases: for the linear system of thermodiffusion, in the quasi-static case of the thermal stresses theory, for the whole system of equations.

Using the above results, Szymaniec (2010) then obtained the global existence and uniqueness of solutions to the Cauchy problem of nonlinear thermodiffusion equations in a solid body.

Recently, Liu and Reissig studied the Cauchy problem for 1D models of thermodiffusion. They explained qualitative properties of solutions and showed which part of the model has a domain influence on well-posedness, propagation of singularities, $L^p$ – $L^q$ decay estimates on the conjugate line and on the diffusion phenomenon.

However, the global existence and asymptotic behavior of solutions for homogeneous, nonhomogeneous and semilinear thermodiffusion equations subject to various boundary conditions are still open.
Some Lemmas on the Semigroup Theory

We shall use the semigroup approaches and the multiplier methods to establish the global existence and exponential stability of solutions for the associated problems. For convenience, we include here some lemmas on the semigroup theory. (See, e.g., Z. Liu and S. Zheng, Semigroups associated with dissipative systems; S. Zheng, Nonlinear Evolution Equations)

Lemma 1.1

Let $A$ be a linear operator with dense domain $D(A)$ in a Hilbert space $H$. If $A$ is dissipative and $0 \in \varrho(A)$, the resolvent set of $A$, then $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $H$. 
Some Lemmas on the Semigroup Theory

Lemma 1.2

Let $S(t) = e^{At}$ be a $C_0$-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if

$$\rho(A) \supseteq \{ i\beta, \beta \in \mathbb{R} \} \equiv i\mathbb{R} \quad (1.12)$$

and

$$\lim_{|\beta| \to +\infty} \| (i\beta I - A)^{-1} \| < +\infty \quad (1.13)$$

hold.
Consider the following initial value problem for the abstract first-order equation:

\[
\begin{cases}
\frac{dy}{dt} + Ay = K, \\
y(0) = y_0,
\end{cases}
\]

(1.14)

where \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions defined in a dense subset \( D(A) \) of a Banach space \( B \). We have the following lemmas.
Some Lemmas on the Semigroup Theory

**Lemma 1.3 (Homogeneous Case)**

Let $K \equiv 0$, suppose that $y_0 \in D(A)$. Then problem (1.14) has a unique classical solution $y(t)$ such that

$$y(t) \in C^1([0, +\infty), B) \cap C([0, +\infty), D(A)).$$

**Lemma 1.4 (Nonhomogeneous Case)**

Let $K = K(x, t)$, suppose that $y_0 \in D(A)$ and $K(t) \in C^1([0, +\infty), B)$. Then problem (1.14) admits a unique global classical solution $y(t)$ such that

$$y(t) \in C^1([0, +\infty), B) \cap C([0, +\infty), D(A)).$$
Lemma 1.5 (Semilinear Case)

Let $K = K(y)$, suppose that $K$ satisfies the global Lipschitz condition, i.e., there is a positive constant $L$ such that for all $y, z \in B$,

\[ \| K(y) - K(z) \|_B \leq L \| y - z \|_B. \]

Then for any $y_0 \in B$ problem (1.14) admits a unique global mild solution $y(t)$ such that $y(t)$ belongs to $C([0, +\infty), B)$. 
Lemma 1.6 (Semilinear Case)

Let $K = K(y)$, suppose that $K \in C^1(B, B)$, i.e., $K = K(y)$ is a nonlinear operator from $B$ into $B$, it is Fréchet differentiable at any $y \in B$, and $K'(y)$ is continuous at $y$. $K(y)$ satisfies the global Lipschitz condition. Then for any $y_0 \in D(A)$ problem (1.14) admits a unique global classical solution.
Homogeneous Problems

We consider the homogeneous equations (1.1)-(1.3) (i.e. $f = g = h \equiv 0$) with the initial condition (1.4) and the boundary condition (1.5):

$$\rho u_{tt} - (\lambda + 2\mu) u_{xx} + \gamma_1 \theta_1 x + \gamma_2 \theta_2 x = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (2.1)$$

$$c \theta_1 t - k \theta_1 xx + \gamma_1 u_{tx} + d \theta_2 t = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (2.2)$$

$$n \theta_2 t - D \theta_2 xx + \gamma_2 u_{tx} + d \theta_1 t = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (2.3)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta_1(x, 0) = \theta_{10}(x), \quad \theta_2(x, 0) = \theta_{20}(x), \quad (2.4)$$

$$u(x, t)|_{x=0,1} = \theta_1(x, t)|_{x=0,1} = \theta_2(x, t)|_{x=0,1} = 0. \quad (2.5)$$
Homogeneous Problems-Global Existence

Using the semigroup approaches, the initial boundary value problem (2.1)-(2.5) is reduced to the following abstract initial value problem for a first-order evolution equation:

\[
\begin{align*}
\frac{dy}{dt} &= A_1 y, \quad \forall t > 0 \\
y|_{t=0} &= y_0 = (u_0, u_1, \theta_{10}, \theta_{20})^T,
\end{align*}
\]

with \( y = (u, u_t, \theta_1, \theta_2)^T \), and

\[
A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{\lambda+2\mu}{\rho} \partial_{xx} & 0 & -\frac{\gamma_1}{\rho} \partial_x & -\frac{\gamma_2}{\rho} \partial_x \\
0 & -\frac{\rho}{kn} \partial_x & -\frac{\rho}{kn} \partial_{xx} & -\frac{\rho}{cd} \partial_{xx} \\
0 & -\frac{d_1-d_2}{nc-d_2} \partial_x & -\frac{d_1-d_2}{nc-d_2} \partial_{xx} & -\frac{d_1-d_2}{nc-d_2} \partial_{xx}
\end{pmatrix}. \tag{2.7}
\]
In order to choose the proper state space for the system, we shall find the static solution first. Thus, we consider the static system associated with (2.1)-(2.3) and (2.5).

\[-(\lambda + 2\mu)\tilde{u}_{xx} + \gamma_1 \tilde{\theta}_1 x + \gamma_2 \tilde{\theta}_2 x = 0, \quad (2.8)\]
\[-k\tilde{\theta}_1 xx = 0, \quad (2.9)\]
\[-D\tilde{\theta}_2 xx = 0, \quad (2.10)\]
\[\tilde{u}|_{x=0,1} = \tilde{\theta}_1|_{x=0,1} = \tilde{\theta}_2|_{x=0,1} = 0. \quad (2.11)\]

Since from (2.11), \(\tilde{\theta}_1, \tilde{\theta}_2\) both vanish at \(x = 0\) and \(x = 1\), it follows from (2.9)-(2.10) that \(\tilde{\theta}_1 = \tilde{\theta}_2 \equiv 0\). Then by (2.8) and the boundary condition for \(\tilde{u}\), we can easily get \(\tilde{u} \equiv 0\). Considering the total energy \(E(t)\) defined in (1.11) and the property of the operator \(A_1\), we can choose the following state space.
Let

\[ H_1 = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \]  

equipped with the norm

\[ \| y \|_{H_1} = \left( \int_0^1 \left[ (\lambda + 2\mu)u_x^2 + \rho u_t^2 + c\theta_1^2 + n\theta_2^2 + 2d\theta_1\theta_2 \right] dx \right)^{\frac{1}{2}}. \]

(2.13)

Instead of dealing with (2.1)-(2.5), we shall consider (2.6) with the domain of the operator \( A_1 \):

\[ D(A_1) = \left( H^2(\Omega) \cap H_0^1(\Omega) \right) \times H_0^1(\Omega) \times \left( H^2(\Omega) \cap H_0^1(\Omega) \right) \times \left( H^2(\Omega) \cap H_0^1(\Omega) \right). \]

(2.14)

It is clear that the operator \( A_1 \) is a densely defined operator from \( D(A_1) \) to \( H_1 \). Furthermore, we have the following lemma.
Lemma 2.1

The operator $A_1$ generates a $C_0$-semigroup $S(t) = e^{A_1 t}$ of contractions on the Hilbert space $H_1$.

Proof of Lemma 2.1

We shall proof it by Lemma 1.1. There are two steps as follows.

- $A_1$ is a dissipative operator, i.e. $(A_1 y, y)_{H_1} \leq 0$.
- $0 \in \varrho(A_1)$ (the resolvent set of $A_1$), i.e. For any $F = (f_1, f_2, f_3, f_4)^T \in H_1$, the equations $A_1 y = F$, exist a unique solution.

We first prove $(A_1 y, y)_{H_1} \leq 0$. Indeed, for any $y \in D(A_1)$, by the definition (2.13) and the boundary condition (2.5), we have
(A_1 y, y)_{H_1} = \int_0^1 \left[ (\lambda + 2\mu)u_{tx}u_x + \rho \left( \frac{\lambda + 2\mu}{\rho} u_{xx} - \frac{\gamma_1}{\rho} \theta_{1x} - \frac{\gamma_2}{\rho} \theta_{2x} \right) u_t ight] dx \\
+ c \left( \frac{d\gamma_2 - n\gamma_1}{nc - d^2} u_{tx} + \frac{kn}{nc - d^2} \theta_{1xx} - \frac{cD}{nc - d^2} \theta_{2xx} \right) \theta_1 \\
+ n \left( \frac{d\gamma_1 - c\gamma_2}{nc - d^2} u_{tx} - \frac{kd}{nc - d^2} \theta_{1xx} + \frac{cD}{nc - d^2} \theta_{2xx} \right) \theta_2 \\
+ d \left( \frac{d\gamma_2 - n\gamma_1}{nc - d^2} u_{tx} + \frac{kn}{nc - d^2} \theta_{1xx} - \frac{dD}{nc - d^2} \theta_{2xx} \right) \theta_2 \\
+ d \left( \frac{d\gamma_1 - c\gamma_2}{nc - d^2} u_{tx} - \frac{kd}{nc - d^2} \theta_{1xx} + \frac{cD}{nc - d^2} \theta_{2xx} \right) \theta_1 \right] dx \\
= - \int_0^1 \left( k\theta_{1x}^2 + D\theta_{2x}^2 \right) dx \leq 0.

which implies that A_1 is a dissipative operator on H_1.
Proof of Lemma 2.1-Continuation

It remains to prove that $0 \in \varrho(A_1)$.

For any $F = (f_1, f_2, f_3, f_4)^T \in H_1$, we consider the equation

$$A_1 y = F,$$  \hspace{1cm} (2.15)

i.e.,

$$u_t = f_1$$  \hspace{1cm} (2.16)

$$\frac{\lambda + 2\mu}{\rho} u_{xx} - \frac{\gamma_1}{\rho} \theta_{1x} - \frac{\gamma_2}{\rho} \theta_{2x} = f_2,$$  \hspace{1cm} (2.17)

$$\frac{d\gamma_2 - n\gamma_1}{nc - d^2} u_{tx} + \frac{kn}{nc - d^2} \theta_{1xx} - \frac{dD}{nc - d^2} \theta_{2xx} = f_3,$$  \hspace{1cm} (2.18)

$$\frac{d\gamma_1 - c\gamma_2}{nc - d^2} u_{tx} - \frac{kd}{nc - d^2} \theta_{1xx} + \frac{cD}{nc - d^2} \theta_{2xx} = f_4.$$  \hspace{1cm} (2.19)

By (2.18) and (2.19), we can get...
Proof of Lemma 2.1-Continuation

\[
\theta_{1xx} = \frac{1}{k}(cf_3 + df_4 - \gamma_1 u_{tx}), \quad (2.20)
\]

\[
\theta_{2xx} = \frac{1}{D}(df_3 + nf_4 - \gamma_2 u_{tx}). \quad (2.21)
\]

We plug \( u_t = f_1 \) obtained from (2.16) into (2.20) to get

\[
\theta_{1xx} = \frac{1}{k}(cf_3 + df_4 - \gamma_1 u_{tx}) \in L^2. \quad (2.22)
\]

By the standard theory in the linear elliptic equations, we have a unique \( \theta_1 \in H^2 \cap H^1_0 \) satisfying (2.20). Using the same method, we can get a unique \( \theta_2 \in H^2 \cap H^1_0 \) satisfying

\[
\theta_{2xx} = \frac{1}{D}(df_3 + nf_4 - \gamma_2 u_{tx}) \in L^2. \quad (2.23)
\]
Proof of Lemma 2.1-Continuation

\[ u_{xx} = \frac{1}{\lambda + 2\mu} (\rho f_2 + \gamma_1 \theta_{1x} + \gamma_2 \theta_{2x}) \in L^2. \quad (2.24) \]

Applying the standard theory in the linear elliptic equations again yields a unique solvability of \( u \in H^2 \cap H^1_0 \) for (2.24). Thus the unique solvability of (2.15) follows and the proof is complete.

Now we are ready to state and prove our main theorem for the global existence of solutions for (2.6) i.e., the homogeneous thermodiffusion equations (2.1)-(2.5).
Theorem 2.1

For any \( y_0 = (u_0, u_1, \theta_{10}, \theta_{20})^T \in D(A_1) \), problem (2.6) has a unique classical solution \( y(t) \) such that

\[
y(t) \in C^1([0, +\infty), H_1) \cap C([0, +\infty), D(A_1)),
\]

i.e., if

\[
u_0 \in H^2(\Omega) \cap H^1_0(\Omega), \quad u_1 \in H^1_0(\Omega),
\]

\[
\theta_{10} \in H^2(\Omega) \cap H^1_0(\Omega), \quad \theta_{20} \in H^2(\Omega) \cap H^1_0(\Omega),
\]
Theorem 2.1-Continuation

The initial boundary value problem for the homogeneous thermodiffusion equations (2.1)-(2.5) has a unique classical solution \((u(t), u_t(t), \theta_1(t), \theta_2(t))\) such that

\[
\begin{align*}
u(t) &\in C^1([0, +\infty), H_0^1(\Omega)) \cap C([0, +\infty), H_1^2(\Omega) \cap H_0^1(\Omega)), \\
u_t(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H_0^1(\Omega)), \\
\theta_1(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H_1^2(\Omega) \cap H_0^1(\Omega)), \\
\theta_2(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H_1^2(\Omega) \cap H_0^1(\Omega)).
\end{align*}
\]

Proof of Theorem 2.1

Using Lemma 1.3 and Lemma 2.1, we can easily complete the proof.
Now we state and prove our exponential stability result for the homogeneous problem (2.1)-(2.5).

**Theorem 2.2**

Suppose that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega), \ u_1 \in H^1_0(\Omega), \theta_{10} \in H^2(\Omega) \cap H^1_0(\Omega), \theta_{20} \in H^2(\Omega) \cap H^1_0(\Omega). \) The total energy \( E(t) \) defined in (1.11) of the initial boundary value problem for the homogeneous thermodiffusion equations (2.1)-(2.5) decays exponentially as time tends to infinity; that is, there exist two positive constants \( M \) and \( \alpha \) independent of the initial data and \( t \), such that

\[
E(t) \leq M E(0) e^{-\alpha t}, \forall t > 0.
\] (2.25)
Proof of Theorem 2.2

We use Lemma 1.2 to prove Theorem 2.2. Sketch of the proof:

- \( \varrho(A) \supseteq \{i\beta, \beta \in \mathbb{R}\} \equiv i\mathbb{R} \). i.e., (1.12).
- \( \lim_{|\beta| \to +\infty} \| (i\beta I - A)^{-1} \| < +\infty \). i.e., (1.13).

These consist of the following steps:

(i) It follows from the fact that \( 0 \in \varrho(A_1) \) and the contraction mapping theorem that for any real number \( \beta \) with \( |\beta| < \| A_1^{-1} \|^{-1} \), the operator \( i\beta I - A_1 = A_1(i\beta A_1^{-1} - I) \) is invertible. Moreover, \( \| (i\beta I - A_1)^{-1} \| \) is a continuous function of \( \beta \) in the interval \( (-\|A_1^{-1}\|^{-1}, \|A_1^{-1}\|^{-1}) \).
Proof of Theorem 2.2-Continuation

(ii) If \( \sup\{\|i\beta l-A_1\|^{-1} \mid |\beta|<\|A_1^{-1}\|^{-1}\}=M<+\infty \), then by the contraction mapping theorem, the operator
\[ i\beta l-A_1=(i\beta_0 l-A_1)(l+i(\beta-\beta_0)(i\beta_0 l-A_1)^{-1}) \] with \( \beta_0<\|A_1^{-1}\|^{-1} \) is invertible for \( |\beta-\beta_0|<\frac{1}{M} \). It turns out that by choosing \( |\beta_0| \) as close to \( \|A_1^{-1}\|^{-1} \) as we can, we conclude that
\[ \{\beta \mid |\beta|<\|A_1^{-1}\|^{-1}+\frac{1}{M}\}\subset \varrho(A_1) \] and \( \|(i\beta l-A_1)^{-1}\| \) is a continuous function of \( \beta \) in the interval \( (-\|A_1^{-1}\|^{-1}-1/M,\|A_1^{-1}\|^{-1}+1/M) \).

(iii) Thus it follows from the argument in (ii) that if (1.12) is not true, then there is \( \omega \in \mathbb{R} \) with \( \|A_1^{-1}\|^{-1}\leq|\omega|<+\infty \) such that
\[ \{i\beta;|\beta|<|\omega|\}\subset \varrho(A_1) \] and \( \sup\{\|(i\beta-A_1)^{-1}\| \mid |\beta|<|\omega|\}=+\infty \). It turns out that there exists a sequence \( \beta_n \in \mathbb{R} \) with \( \beta_n \to \omega, \; |\beta_n|<|\omega| \) and a sequence of complex vector functions \( y_n \in D(A_1) \) with \( \|y_n\|_{H_1}=1 \) such that
Proof of Theorem 2.2-Continuation

\[ \| (i \beta_n l - A_1) y_n \|_{H_1} \to 0, \tag{2.26} \]

as \( n \to +\infty \), i.e.,

\[ i \beta_n u_n - u_{nt} \to 0 \quad \text{in } H_1^1, \tag{2.27} \]

\[ -\frac{\lambda + 2\mu}{\rho} u_{nx} + i \beta_n u_{nt} + \frac{\gamma_1}{\rho} \theta_{1nx} + \frac{\gamma_2}{\rho} \theta_{2nx} \to 0 \quad \text{in } L^2, \tag{2.28} \]

\[ n \gamma_1 - \frac{d}{nc-d^2} u_{ntx} + i \beta_n \theta_{1n} - \frac{kn}{nc-d^2} \theta_{1nxx} + \frac{D}{nc-d^2} \theta_{2nxx} \to 0 \quad \text{in } L^2, \tag{2.29} \]

\[ \frac{c}{nc-d^2} u_{ntx} + \frac{k}{nc-d^2} \theta_{1nxx} + i \beta n \theta_{2n} - \frac{cD}{nc-d^2} \theta_{2nxx} \to 0 \quad \text{in } L^2. \tag{2.30} \]

Taking the inner product of \((i \beta_n l - A_1) y_n\) with \( y_n \) in \( H_1 \) and then taking its real part yields as \( n \to +\infty \)

\[ Re((i \beta_n l - A_1) y_n, y_n)_{H_1} = (k \| \theta_{1nx} \|^2 + D \| \theta_{2nx} \|^2) \to 0 \quad \text{in } L^2. \tag{2.31} \]
Proof of Theorem 2.2-Continuation

Thus we can derive from (2.29), (2.30), (2.31) and the Poincaré inequality that as $n \to +\infty$

\[ k\theta_{1nx} - \gamma_1u_{ntx} \to 0 \quad \text{in } L^2, \quad (2.32) \]
\[ D\theta_{2nx} - \gamma_2u_{ntx} \to 0 \quad \text{in } L^2. \quad (2.33) \]

Integrating (2.32), (2.33) from 0 to $x$ yields as $n \to +\infty$

\[ k\theta_{1nx} - k\theta_{1nx}(0) - \gamma_1u_{nt}(x) \to 0 \quad \text{in } L^2, \quad (2.34) \]
\[ D\theta_{2nx} - D\theta_{2nx}(0) - \gamma_2u_{nt}(x) \to 0 \quad \text{in } L^2. \quad (2.35) \]

Combining (2.34) and (2.35) with (2.31) yields as $n \to +\infty$

\[ k\theta_{1nx}(0) + \gamma_1u_{nt}(x) \to 0 \quad \text{in } L^2, \quad (2.36) \]
\[ D\theta_{2nx}(0) + \gamma_2u_{nt}(x) \to 0 \quad \text{in } L^2. \quad (2.37) \]
Proof of Theorem 2.2-Continuation

By \( \| y_n \|_{H_1} = 1 \) and (2.27), we get that \( \| u_{ntx} \| \) is uniformly bounded with respect to \( n \). Thus it follows from (2.32) and (2.33) that \( \| \theta_{1nxx} \| \), and \( \| \theta_{2nxx} \| \) are both uniformly bounded. By the Gagliardo-Nirenberg inequality, we obtain that as \( n \to +\infty \)

\[
|\theta_{1n}(0)| \leq \| \theta_{1n} \|_{L^\infty} \leq C_1 \| \theta_{1nxx} \| \frac{1}{2} \| \theta_{1nx} \| \frac{1}{2} + C_2 \| \theta_{1nx} \| \to 0, \tag{2.38}
\]

\[
|\theta_{2n}(0)| \leq \| \theta_{2n} \|_{L^\infty} \leq C_3 \| \theta_{2nxx} \| \frac{1}{2} \| \theta_{2nx} \| \frac{1}{2} + C_4 \| \theta_{2nx} \| \to 0. \tag{2.39}
\]

Combining (2.38) with (2.36) yields as \( n \to +\infty \)

\[
\| u_{nt} \| \to 0 \quad \text{in } L^2. \tag{2.40}
\]

Taking the inner product of (2.28) with \( u_n \) in \( L^2 \) and integrating by parts also yields as \( n \to +\infty \)
Proof of Theorem 2.2-Continuation

We now prove (1.13) by a contradiction argument again. Suppose that (1.13) is not true. Then there exists a sequence \( \beta_n \to +\infty \) and a sequence of complex vector functions \( y_n \in D(A_1) \) with unit norm in \( H_1 \) such that (2.26) holds. Again we have (2.31). The remaining proof is more delicate than that in (iii) because \( \beta_n \to +\infty \) now. Dividing (2.29) and (2.30) by \( \beta_n \) and using the Poincaré inequality, we get as \( n \to +\infty \)

\[
\frac{1}{\beta_n}(k\theta_1 nxx - \gamma_1 u_{ntx}) \to 0 \quad \text{in } L^2, \tag{2.42}
\]

\[
\frac{1}{\beta_n}(D\theta_2 nxx - \gamma_2 u_{ntx}) \to 0 \quad \text{in } L^2. \tag{2.43}
\]

Dividing (2.27) by \( \beta_n \) and using (2.42)-(2.43), we obtain as \( n \to +\infty \)
Proof of Theorem 2.2-Continuation

\[ \frac{k}{\beta_n} \theta_{1nxx} - i \gamma_1 u_{nx} \to 0 \quad \text{in } L^2, \quad (2.44) \]

\[ \frac{D}{\beta_n} \theta_{2nxx} - i \gamma_2 u_{nx} \to 0 \quad \text{in } L^2. \quad (2.45) \]

By the condition \( \| y_n \| = 1 \), we can get \( \| u_{nx} \| \) is bounded. Thus it follows from (2.44) and (2.45) that \( \| \frac{\theta_{1nxx}}{\beta_n} \| \) and \( \| \frac{\theta_{2nxx}}{\beta_n} \| \) are both bounded. Taking the inner product of (2.44) with \( u_{nx} \) in \( L^2 \) yields as \( n \to +\infty \)

\[ (\frac{k\theta_{1nxx}}{\beta_n}, u_{nx}) = \frac{k\theta_{1nxu_{nx}}}{\beta_n} \bigg|_{x=1} - \frac{k\theta_{1nxu_{nx}}}{\beta_n} \bigg|_{x=0} - (\frac{k\theta_{1nx}}{\beta_n}, u_{nxx}). \quad (2.46) \]

By integration by parts, we have

\[ \frac{k\theta_{1nxx}}{\beta_n} u_{nx} \bigg|_{x=1} - \frac{k\theta_{1nxu_{nx}}}{\beta_n} \bigg|_{x=0} - (\frac{k\theta_{1nx}}{\beta_n}, u_{nxx}). \quad (2.47) \]
Proof of Theorem 2.2-Continuation

Dividing (2.28) by $\beta_n$ and using (2.31) and the fact that $\| u_{nt} \|$ is bounded, we deduce that $\frac{u_{nxx}}{\beta_n}$ is bounded. Thus it follows from (2.31) and the Cauchy-Schwartz inequality that as $n \to +\infty$

$$(\frac{k\theta_{1nx}}{\beta_n}, u_{nxx}) \to 0.$$  

(2.48)

By the Gagliardo-Nirenberg inequality, we have

$$\| \frac{\theta_{1nx}}{1} \|_{L^\infty} \leq C_1 \| \theta_{1nx} \| \frac{1}{|\beta_n|^2} \| \theta_{1nxx} \| \frac{1}{|\beta_n|^2} + C_2 \| \theta_{1nx} \| \frac{1}{|\beta_n|^2} \to 0, \text{ as } n \to +\infty$$  

(2.49)

and

$$\| \frac{u_{nx}}{1} \|_{L^\infty} \leq C_3 \| u_{nx} \| \frac{1}{|\beta_n|^2} \| u_{nxx} \| \frac{1}{|\beta_n|^2} + C_4 \| u_{nx} \| \frac{1}{|\beta_n|^2} \leq C.$$  

(2.50)

with $C$ being a positive constant independent of $n$. Then it turns out from (2.49) and (2.50) that
Proof of Theorem 2.2-Continuation

\[ \| \frac{k u_{nx} \theta_{1nx}}{\beta_n} \|_{L^\infty} \leq k \frac{\| u_{nx} \|_{L^\infty}}{\| \theta_{1nx} \|_{L^\infty}} \rightarrow 0, \text{ as } n \rightarrow +\infty \quad (2.51) \]

which, combined with (2.46)-(2.48), yields as \( n \rightarrow +\infty \)

\[ \| u_{nx} \| \rightarrow 0. \quad (2.52) \]

Thus by (2.27), we get as \( n \rightarrow +\infty \)

\[ \frac{u_{ntx}}{\beta_n} \rightarrow 0 \quad \text{in } L^2. \quad (2.53) \]

Taking the inner product of (2.28) with \( u_{nt} \) in \( L^2 \) and dividing the result by \( \beta_n \), we obtain that as \( n \rightarrow +\infty \)

\[ i \| u_{nt} \|^2 + \frac{\lambda + 2\mu}{\rho} (u_{nx}, \frac{u_{ntx}}{\beta_n}) \rightarrow 0 \quad \text{in } L^2. \quad (2.54) \]
Proof of Theorem 2.2-Continuation

Therefore, from (2.52)-(2.54), it follows that as \( n \to +\infty \)

\[
u_{nt} \to 0 \quad \text{in } L^2. \tag{2.55}\]

Thus (2.55), (2.52), and (2.31) contradict \( \| y_n \|_{H_1} = 1 \).

The proof of Theorem 2.2 is complete. \( \square \)
Nonhomogeneous Problems

When \( f = f(x, t) \), \( g = g(x, t) \), \( h = h(x, t) \), (1.1)-(1.3) is a system of nonhomogeneous thermodiffusion equations, we consider the following system:

\[
\begin{align*}
\rho u_{tt} - (\lambda + 2\mu) u_{xx} + \gamma_1 \theta_1 x + \gamma_2 \theta_2 x &= f(x, t), \quad \text{in } \Omega \times \mathbb{R}^+ , \\
\rho \theta_1 t - k \theta_1 xx + \gamma_1 u tx + d \theta_2 t &= g(x, t), \quad \text{in } \Omega \times \mathbb{R}^+ , \\
\rho \theta_2 t - D \theta_2 xx + \gamma_2 u tx + d \theta_1 t &= h(x, t), \quad \text{in } \Omega \times \mathbb{R}^+ , \\
\end{align*}
\]

(3.1) \hspace{1cm} (3.2) \hspace{1cm} (3.3)

\[
\begin{align*}
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta_1(x, 0) = \theta_{10}(x), \quad \theta_2(x, 0) = \theta_{20}(x), \\
\end{align*}
\]

(3.4)

\[
\begin{align*}
u(x, t)|_{x=0,1} = \theta_1(x, t)|_{x=0,1} = \theta_2(x, t)|_{x=0,1} = 0,
\end{align*}
\]

(3.5)
The main result on the global existence and uniqueness for the problem (3.1)-(3.5) is formulated here.

**Theorem 3.1**

Suppose that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $u_1 \in H^1_0(\Omega)$, $\theta_{10} \in H^2(\Omega) \cap H^1_0(\Omega)$, $\theta_{20} \in H^2(\Omega) \cap H^1_0(\Omega)$ and $f(x, \cdot), g(x, \cdot), h(x, \cdot) \in C^1([0, +\infty), L^2(\Omega))$. Then the problem (3.1)-(3.5) has a unique classical solution $(u(t), u_t(t), \theta_1(t), \theta_2(t))$ such that

- $u(t) \in C^1([0, +\infty), H^1_0(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega))$,
- $u_t(t) \in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^1_0(\Omega))$,
- $\theta_1(t) \in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega))$,
- $\theta_2(t) \in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega))$. 
Proof of Theorem 3.1

We use Lemma 1.4 to prove this theorem. Using the semigroup approaches, the initial boundary value problem (3.1)-(3.5) is reduced to the following abstract initial value problem for a first-order evolution equation:

\[
\begin{aligned}
\frac{dy}{dt} + B_1 y &= F, \quad \forall t > 0 \\
y \big|_{t=0} &= y_0 = (u_0, u_1, \theta_{10}, \theta_{20})^T
\end{aligned}
\]

(3.6)

with \( y = (u, u_t, \theta_1, \theta_2)^T \), \( F = (0, f(x, t), g(x, t), h(x, t))^T \), and \( B_1 = -A_1 \). \( A_1 \) is defined in (2.7).
Proof of Theorem 3.1-Continuation

Using the same method as in Section 2, we can choose the same state space $H_1$ as defined in (2.12) and equip $H_1$ with the same norm as in (2.13). Instead of dealing with (3.1)-(3.5), we shall consider (3.6) with the domain of operator $B_1$:

$$D(B_1) = D(A_1). \quad D(A_1) \text{ is defined in (2.14).}$$

From the proof in Section 2, we can conclude that the operator $B_1$ is the infinitesimal generator of a $C_0$-semigroup of contractions defined in the dense subject $D(B_1)$ of the Hilbert space $H_1$. From our assumption, we can get $F \in C^1([0, +\infty), H_1), \ u_0 \in D(B_1)$.

Thus by Lemma 1.4, the proof of Theorem 3.1 is completed. \(\square\)
Now we shall use the multiplier methods to study the asymptotic behavior of solutions for the problem (3.1)-(3.5). For this purpose, we first establish several lemmas.

**Lemma 3.1**

Suppose that $y(t) \in C^1(\mathbb{R}^+)\), $y(t) \geq 0$, $\forall t > 0$, and satisfies

$$y'(t) \leq -C_0y(t) + \lambda(t), \ \forall t > 0,$$  \hspace{1cm} (3.7)

where $0 \leq \lambda(t) \in L^1(\mathbb{R}^+)$ and $C_0$ is a positive constant. Then we have

$$\lim_{t \rightarrow +\infty} y(t) = 0,$$  \hspace{1cm} (3.8)
**Lemma 3.1-Continuation**

Furthermore,

(1) if \( \lambda(t) \leq C_1 e^{-\alpha_0 t}, \forall t > 0, \) with \( C_1 > 0, \alpha_0 > 0 \) being constants, then

\[
y(t) \leq C_2 e^{-\alpha t}, \forall t > 0, \tag{3.9}
\]

with \( C_2 > 0, \alpha > 0 \) being constants.

(2) if \( \lambda(t) \leq C_3 (1 + t)^{-p}, \forall t > 0, \) with \( p > 1, C_3 > 0 \) being constants, then

\[
y(t) \leq C_4 (1 + t)^{-p+1}, \forall t > 0, \tag{3.10}
\]

with a constant \( C_4 > 0. \)
Proof of Lemma 3.1

Multiplying (3.7) by $e^{C_0 t}$ and integrating the resulting inequality, we have

$$y(t) \leq y(0)e^{-C_0 t} + e^{-C_0 t} \int_0^t \lambda(s)e^{C_0 s} ds. \tag{3.11}$$

Noting that $\lambda(t) \in L^1(\mathbb{R}^+)$, we derive

$$e^{-C_0 t} \int_0^t \lambda(s)e^{C_0 s} ds$$

$$= \int_0^{t/2} \lambda(s)e^{-C_0 (t-s)} ds + \int_{t/2}^t \lambda(s)e^{-C_0 (t-s)} ds$$
Proof of Lemma 3.1-Continuation

\[ \leq e^{-\frac{c_0}{2}t} \int_0^{+\infty} \lambda(s)ds + \int_{\frac{t}{2}}^{t} \lambda(s)ds \rightarrow 0, \text{ as } t \rightarrow +\infty. \quad (3.12) \]

Thus (3.8) follows from (3.11) and (3.12).

(1) If \( \lambda(t) \leq C_1 e^{-\alpha_0 t}, \forall t > 0 \), then it follows from (3.11)-(3.12) that

\[ y(t) \leq y(0)e^{-c_0 t} + C_1'e^{-\frac{c_0}{2} t} + C_1 \int_{\frac{t}{2}}^{t} e^{-\alpha_0 s}ds \]

\[ \leq y(0)e^{-c_0 t} + C_1'e^{-\frac{c_0}{2} t} + \frac{C_1}{\alpha} e^{-\frac{\alpha_0}{2} t} \]

\[ \leq C_2 e^{-\alpha t}, \]
Proof of Lemma 3.1-Continuation

with $C_1' = \int_0^{+\infty} \lambda(s)ds$, $C_2 = \max\{y(0), C_1', \frac{c_1}{\alpha_0}\} > 0$ and $\alpha = \frac{1}{2} \min\{C_0, \alpha_0\} > 0$. This proves (3.9).

(2) If $\lambda(t) \leq C_3(1 + t)^{-p}$, similarly to case (1), it follows from (3.11)-(3.12) that

$$y(t) \leq y(0)e^{-c_0 t} + C_1' e^{-\frac{c_0}{2} t} + C_3 \int_{\frac{t}{2}}^{t} \frac{1}{(1 + s)^p} ds$$

$$\leq y(0)e^{-c_0 t} + C_1' e^{-\frac{c_0}{2} t} + \frac{C_3}{p - 1} (1 + \frac{t}{2})^{-p+1}$$

$$\leq C_4 (1 + t)^{-p+1}, \forall t > 0,$$

for some constant $C_4 = C_4(y(0), C_1', C_3, p) > 0$ depending on $y(0), C_1', C_3$ and $p$. This gives (3.10) and hence the proof is complete.□
Lemma 3.2

Let \((u, u_t, \theta_1, \theta_2)\) be the solution of the problem (3.1)-(3.5). Then the total energy \(E(t)\) defined by (1.11), satisfies the following inequality for all \(\varepsilon_1 > 0\),

\[
E'(t) \leq -k \parallel \theta_{1x} \parallel^2 - D \parallel \theta_{2x} \parallel^2 + \varepsilon_1 \parallel u_t \parallel^2 + C(\varepsilon_1) \parallel f \parallel^2 \\
+ C_1(\parallel g \parallel^2 + \parallel h \parallel^2)
\]

(3.13)

where \(C_1\) is a positive constant.
Proof of Lemma 3.2

Multiplying (3.1), (3.2), (3.3) by $u_t$, $\theta_1$, $\theta_2$ respectively, integrating the results over $[0, 1]$, using Young's inequality, Poincaré inequality and (3.5), we can get for all $\varepsilon_1 > 0$,

$$E'(t) = \int_0^1 (fu_t + g\theta_1 + h\theta_2)dx - 2k \| \theta_{1x} \|^2 - 2D \| \theta_{2x} \|^2$$

$$\leq \varepsilon_1 \| u_t \|^2 + C(\varepsilon_1) \| f \|^2 + k \| \theta_{1x} \|^2 + C(k) \| g \|^2 + D \| \theta_{2x} \|^2 + C(D) \| h \|^2 - 2k \| \theta_{1x} \|^2 - 2D \| \theta_{2x} \|^2,$$

where $C(\varepsilon_1)$, $C(k)$ and $C(D)$ are some constants depending on $\varepsilon_1$, $k$ and $D$, respectively.

Then we complete the proof of Lemma 3.2. \qed
Now we are going to construct a Lyapunov function $L$ equivalent to $E$, with which we can show the desired result. Let

$$L_1(t) = \rho \int_0^1 uu_t \, dx.$$ (3.14)

**Lemma 3.3**

Let $(u, u_t, \theta_1, \theta_2)$ be the solution of the problem (3.1)-(3.5). Then we have,

$$L'_1(t) \leq \rho \| u_t \|^2 - \frac{\lambda + 2\mu}{2} \| u_x \|^2 + C_2(\| \theta_{1x} \|^2 + \| \theta_{2x} \|^2 + \| f \|^2),$$ (3.15)

where $C_2$ is a positive constant.
Proof of Lemma 3.3

By (3.1)-(3.3), Young’s inequality and Poincaré inequality, we can get for all $\varepsilon_2 > 0$,

$$L'_1(t) = \rho \int_0^1 u_t^2 \, dx + \rho \int_0^1 uu_{tt} \, dx$$

$$= \rho \int_0^1 u_t^2 \, dx + \int_0^1 u[(\lambda + 2\mu)u_{xx} - \gamma_1 \theta_1 x - \gamma_2 \theta_2 x + f] \, dx$$

$$\leq \rho \| u_t \|^2 - (\lambda + 2\mu) \| u_x \|^2 + \varepsilon_2 \| u_x \|^2 + C(\varepsilon_2) \| \theta_1 x \|^2$$

$$+ \varepsilon_2 \| u_x \|^2 + C(\varepsilon_2) \| \theta_2 x \|^2 + \varepsilon_2 \| u_x \|^2 + C(\varepsilon_2) \| f \|^2.$$ 

Taking $\varepsilon_2 = \frac{(\lambda + 2\mu)}{6}$, we complete the proof of Lemma 3.3. □
Let

$$L_2(t) = d \rho c \int_0^1 \int_0^x \theta_1 \, dy \, u_t \, dx + c \rho n \int_0^1 \int_0^x \theta_2 \, dy \, u_t \, dx. \quad (3.16)$$

**Lemma 3.4**

Let \((u, u_t, \theta_1, \theta_2)\) be the solution of the problem (3.1)-(3.5). Then for all \(\varepsilon_3 > 0\) and \(\varepsilon_4 > 0\), we have,

$$L'_2(t) \leq \varepsilon_3 \| u_x \|^2 + C(\varepsilon_3) \| \theta_1 x \|^2 + \varepsilon_4 \| u_x \|^2 + C(\varepsilon_4) \| \theta_2 x \|^2 - \frac{c \rho \gamma^2}{2} \| u_t \|^2 + C_3(\| f \|^2 + \| h \|^2), \quad (3.17)$$

where \(C_3\) is a positive constant.
Proof of Lemma 3.4

By (3.1)-(3.3), Young’s inequality and Poincaré inequality, we can get for all $\varepsilon_3 > 0$, $\varepsilon_4 > 0$ and $\varepsilon_5 > 0$,

\[
L'_2(t) = d\rho c \int_0^1 \left( \int_0^x \theta_1 dy \right) u_t dx + d\rho c \int_0^1 \left( \int_0^x \theta_1 dy \right) u_{tt} dx
\]
\[
+c\rho n \int_0^1 \left( \int_0^x \theta_2 dy \right) u_t dx + c\rho n \int_0^1 \left( \int_0^x \theta_2 dy \right) u_{tt} dx
\]
\[
\leq \varepsilon_3 \| u_x \|^2 + C(\varepsilon_3) \| \theta_1x \|^2 + \varepsilon_4 \| u_x \|^2 + C(\varepsilon_4) \| \theta_2x \|^2
\]
\[
+ \varepsilon_5 \| u_t \|^2 + C(\varepsilon_5) \| h \|^2 + \varepsilon_5 \| u_t \|^2 + C(\varepsilon_5) \| \theta_2x \|^2
\]
\[
- c\rho \gamma_2 \| u_t \|^2 + C_7 \| f \|^2.
\]

(3.18)

Taking $\varepsilon_5 = \frac{c\rho \gamma_2}{4}$ in (3.18), we can complete the proof of Lemma 3.4. □
We are now ready to state and prove our main theorem on asymptotic behavior of global solutions for the problem (3.1)-(3.5).

**Theorem 3.2**

Suppose that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), \( u_1 \in H^1_0(\Omega) \), \( \theta_{10} \in H^2(\Omega) \cap H^1_0(\Omega) \), \( \theta_{20} \in H^2(\Omega) \cap H^1_0(\Omega) \) and
\( f(x, \cdot), \ g(x, \cdot), \ h(x, \cdot) \in C^1([0, +\infty), L^2(\Omega)) \),
\( \int_0^{+\infty} (\| f \|^2 + \| g \|^2 + \| h \|^2) dt < +\infty \), \( (u(t), u_t(t), \theta_1(t), \theta_2(t)) \)
is the global solution of problem (3.1)-(3.5). Then the total energy \( E(t) \) defined by (1.11) satisfies

\[
\lim_{t \to +\infty} E(t) = 0. \tag{3.19}
\]
Theorem 3.2-Continuation

Furthermore

(1) if

\[ \|f\|^2 + \|g\|^2 + \|h\|^2 \leq C_0 e^{-\alpha_0 t}, \quad \forall t > 0, \]  

(3.20)

with \( C_0 > 0 \) and \( \alpha_0 > 0 \) being constants, then there exist positive constants \( M, \alpha \) such that \( E(t) \) satisfies

\[ E(t) \leq M e^{-\alpha t}, \quad \forall t > 0. \]  

(3.21)

(2) if

\[ \|f\|^2 + \|g\|^2 + \|h\|^2 \leq C'(1+t)^{-p}, \quad \forall t > 0, \]  

(3.22)

with constants \( C' > 0, \ p > 1 \), then there exists a constant \( C^* > 0 \) such that

\[ E(t) \leq C^*(1+t)^{-p+1}, \quad \forall t > 0. \]  

(3.23)
Proof of Theorem 3.2

For $N > 1$, let

$$L(t) = NE(t) + \frac{c\gamma^2}{8} L_1(t) + L_2(t).$$

(3.24)

By combining (3.13)-(3.18), we obtain

$$L'(t) \leq -\left(\frac{\lambda + 2\mu}{2} \cdot \frac{c\gamma^2}{8} - \varepsilon_3 - \varepsilon_4\right)\|u_x\|^2 - \left(\frac{c\rho\gamma^2}{2} - \frac{c\rho\gamma^2}{8} - \varepsilon_1 N\right)\|u_t\|^2$$

$$- (Nk - C(\varepsilon_3) - C_2 \cdot \frac{c\gamma^2}{8})\|\theta_1\|^2 - (ND - C(\varepsilon_4) - C_2 \cdot \frac{c\gamma^2}{8})\|\theta_2\|^2$$

$$+ (C(\varepsilon_1)N + C_2 \cdot \frac{c\gamma^2}{8} + C_3)\|f\|^2 + C_1 N\|g\|^2 + (C_1 N + C_3)\|h\|^2.$$

At this point,
Proof of Theorem 3.2-Continuation

At this point, we choose $\varepsilon_3, \varepsilon_4$ small enough so that

$$\kappa_1 := \left( \frac{\lambda + 2\mu}{2} \cdot \frac{c\gamma_2}{8} - \varepsilon_3 - \varepsilon_4 \right) > 0,$$

then $N$ large enough so that

$$\kappa_2 := (Nk - C(\varepsilon_3) - C_2 \cdot \frac{c\gamma_2}{8}) > 0,$$

$$\kappa_3 := (ND - C(\varepsilon_4) - C_2 \cdot \frac{c\gamma_2}{8}) > 0.$$

Next we choose $\varepsilon_1$ small enough so that

$$\kappa_4 := \left( \frac{c\rho\gamma_2}{2} - \frac{c\rho\gamma_2}{8} - \varepsilon_1 N \right) > 0.$$
Proof of Theorem 3.2-Continuation

Using (1.11) and the assumptions, we arrive at

\[ L'(t) \leq -\kappa_1 \| u_x \|^2 - \kappa_2 \| \theta_{1x} \|^2 - \kappa_3 \| \theta_{2x} \|^2 - \kappa_4 \| u_t \|^2 
+ C_4 (\| f \|^2 + \| g \|^2 + \| h \|^2) \]
\[ \leq -\kappa E(t) + C_4 (\| f \|^2 + \| g \|^2 + \| h \|^2), \tag{3.25} \]

for some constants \( \kappa, \ C_4 > 0 \). On the other hand, we find that for some constants \( C_5 > 0 \) and \( C_6 > 0 \),

\[ |L(t) - NE(t)| \leq \left| \frac{c^2\gamma^2}{8} L_1(t) \right| + |L_2(t)| \]
\[ \leq \left| \frac{c^2\rho}{8} \int_0^l |uu_t| dx + c\rho \int_0^l \left( \int_0^x \theta_1 dy \right) u_t |dx \right| + c\rho \int_0^l \left( \int_0^x \theta_2 dy \right) u_t |dx \]
\[ \leq C_5 (\| u_t \|^2 + \| u_x \|^2 + \| \theta_1 \|^2 + \| \theta_2 \|^2) \]
\[ \leq C_6 E(t). \]
Proof of Theorem 3.2-Continuation

Therefore, we can choose $N$ even large (if needed) so that $L(t)$ is equivalent to $E(t)$, i.e.,

$$C_7 E(t) \leq L(t) \leq C_8 E(t),$$  \hspace{1cm} (3.26)

for same constants $C_7 > 0$, $C_8 > 0$. Hence, (3.25) and (3.26) lead to

$$L'(t) \leq -\kappa C_8^{-1} L(t) + C_4 (\| f \|^2 + \| g \|^2 + \| h \|^2).$$  \hspace{1cm} (3.27)

Applying Lemma 3.1 with $y(t)=L(t)$ and $\lambda(t)=C_4 (\| f \|^2 + \| g \|^2 + \| h \|^2)$, we can complete the proof. \hfill \Box
Now we consider the semilinear problem, i.e.,
\[ f = f(u, u_t, \theta_1, \theta_2), \quad g = g(u, u_t, \theta_1, \theta_2), \quad h = h(u, u_t, \theta_1, \theta_2). \]
We consider the following system:

\[
\begin{align*}
\rho u_{tt} - (\lambda + 2\mu)u_{xx} + \gamma_1 \theta_1 x + \gamma_2 \theta_2 x &= f(u, u_t, \theta_1, \theta_2), \quad \text{in } \Omega \times \mathbb{R}^+, (4.1) \\
c\theta_{1t} - k\theta_{1xx} + \gamma_1 u_{tx} + d\theta_{2t} &= g(u, u_t, \theta_1, \theta_2), \quad \text{in } \Omega \times \mathbb{R}^+, (4.2) \\
n\theta_{2t} - D\theta_{2xx} + \gamma_2 u_{tx} + d\theta_{1t} &= h(u, u_t, \theta_1, \theta_2), \quad \text{in } \Omega \times \mathbb{R}^+, (4.3)
\end{align*}
\]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta_1(x, 0) = \theta_{10}(x), \quad \theta_2(x, 0) = \theta_{20}(x), \quad (4.4) \]

\[ u(x, t)|_{x=0,1} = \theta_1(x, t)|_{x=0,1} = \theta_2(x, t)|_{x=0,1} = 0, \quad (4.5) \]
We formulate the main results on the global existence and uniqueness for the problem (4.1)-(4.5) here.

**Theorem 4.1**

Suppose that

\[ f(z_1, z_2, z_3, z_4), \ g(z_1, z_2, z_3, z_4), \ h(z_1, z_2, z_3, z_4) \in C^1(\mathbb{R}^4, \mathbb{R}) \]

and \( \nabla f, \ \nabla g, \ \nabla h \) are uniformly bounded. Then for any

\( u_0 \in H^1_0(\Omega), \ u_1 \in L^2(\Omega), \ \theta_{10} \in L^2(\Omega), \ \theta_{20} \in L^2(\Omega) \),

problem (4.1)-(4.5) has a unique mild solution \((u(t), u_t(t), \theta_1(t), \theta_2(t))\) such that

\[
\begin{align*}
    u(t) &\in C^1([0, +\infty), H^1_0(\Omega)), \\
    u_t(t) &\in C^1([0, +\infty), L^2(\Omega)), \\
    \theta_1(t) &\in C^1([0, +\infty), L^2(\Omega)), \\
    \theta_2(t) &\in C^1([0, +\infty), L^2(\Omega)).
\end{align*}
\]
Theorem 4.1-Continuation

Furthermore, if \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega), \ u_1 \in H^1_0(\Omega), \ \theta_{10} \in H^2(\Omega) \cap H^1_0(\Omega), \ \theta_{20} \in H^2(\Omega) \cap H^1_0(\Omega) \), then problem (4.1)-(4.3), (1.4) and (1.5) has a unique classical solution \((u(t), u_t(t), \theta_1(t), \theta_2(t))\) such that

\[
\begin{align*}
    u(t) &\in C^1([0, +\infty), H^1_0(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)), \quad (4.8) \\
    u_t(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^1_0(\Omega)), \quad (4.9) \\
    \theta_1(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)), \quad (4.10) \\
    \theta_2(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)). \quad (4.11)
\end{align*}
\]
Proof of Theorem 4.1

Using the semigroup approaches, the initial boundary value problem (4.1)-(4.5) is reduced to the same abstract initial value problem for a first-order evolution equation as defined in (3.6) with \( F = F(0, f(u, u_t, \theta_1, \theta_2), g(u, u_t, \theta_1, \theta_2), h(u, u_t, \theta_1, \theta_2)) \). Choose the same state space \( H_1 \) as defined in (2.12) and equip \( H_1 \) with the same norm as defined in (2.13). Instead of dealing with (4.1)-(4.5), we shall consider (3.6) with the domain of operator \( B_1 : D(B_1) = D(A_1) \). \( D(A_1) \) is defined in (2.14).

From the proof in Section 2 and the semigroup theory, we can get that the operator \( B_1 \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions defined in the dense subject \( D(B_1) \) of the Hilbert space \( H_1 \).
Proof of Theorem 4.1-Continuation

By the assumptions, $F$ is a nonlinear operator from $H_1$ to $H_1$, $F \in C^1(H_1, H_1)$, and $F$ satisfies the global Lipschitz condition. Thus, by Lemma 1.5, when

$u_0 \in H^1_0(\Omega), \ u_1 \in L^2(\Omega), \ \theta_{10} \in L^2(\Omega), \ \theta_{20} \in L^2(\Omega)$, i.e., $y_0 \in H_1$, problem (4.1)-(4.5) admits a unique mild solution $(u, u_t, \theta_1, \theta_2)$ such that (4.6) and (4.7) are true.

When $u_0 \in H^2(\Omega) \cap H^1_0(\Omega), \ u_1 \in H^1_0(\Omega), \ \theta_{10} \in H^2(\Omega) \cap H^1_0(\Omega), \ \theta_{20} \in H^2(\Omega) \cap H^1_0(\Omega)$, i.e., $y_0 \in D(B_1)$, by Lemma 1.5, problem (4.1)-(4.5) admits a unique classical solution $(u, u_t, \theta_1, \theta_2)$ such that (4.8)-(4.11) are true. Thus the proof is complete.
We can use the same methods to consider the thermodiffusion equations (1.1)-(1.3) subject to the initial condition (1.4) and other boundary conditions. The main difference in the proof between different boundary conditions are choosing different state spaces and different operator domains.
Now we take the initial boundary value problem (1.1)-(1.4) and (1.6) for example to illustrate these difference.

\[ \rho u_{tt} - (\lambda + 2\mu)u_{xx} + \gamma_1 \theta_{1x} + \gamma_2 \theta_{2x} = f, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (5.1) \]
\[ c \theta_{1t} - k \theta_{1xx} + \gamma_1 u_{tx} + d \theta_{2t} = g, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (5.2) \]
\[ n \theta_{2t} - D \theta_{2xx} + \gamma_2 u_{tx} + d \theta_{1t} = h, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (5.3) \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta_1(x, 0) = \theta_{10}(x), \quad \theta_2(x, 0) = \theta_{20}(x), \quad (5.4) \]
\[ u(x, t)|_{x=0,1} = \theta_{1x}(x, t)|_{x=0,1} = \theta_{2x}(x, t)|_{x=0,1} = 0, \quad (5.5) \]
In order to choose the proper state space for (5.1)-(5.6), we shall consider the static system associated with (5.1)-(5.3) and (5.6).

\[-(\lambda + 2\mu)\ddot{u}_{xx} + \gamma_1 \ddot{\theta}_1 + \gamma_2 \ddot{\theta}_2 = 0, \quad (5.6)\]
\[-k \ddot{\theta}_1 = 0, \quad (5.7)\]
\[-D \ddot{\theta}_2 = 0, \quad (5.8)\]
\[\ddot{u}|_{x=0,1} = \ddot{\theta}_1|_{x=0,1} = \ddot{\theta}_2|_{x=0,1} = 0. \quad (5.9)\]

Since \(\ddot{\theta}_1\) and \(\ddot{\theta}_2\) both vanish at \(x = 0\) and \(x = 1\), it follows from the equations (5.7)-(5.8) that \(\ddot{\theta}_1 \equiv C_2\), \(\ddot{\theta}_2 \equiv C_2\) are nonzero solutions of (5.6)-(5.9) for any constant numbers \(C_1\), \(C_2\).

Therefore, we shall impose \(\int_0^1 \theta_1(x)dx = 0\), \(\int_0^1 \theta_2(x)dx = 0\) to force \(\ddot{\theta}_1 = \ddot{\theta}_2 \equiv 0\).
Considering the total energy (1.11) and the property of operator $A$, we can choose the following state space and the domain of operator $A$ for problem (5.1)-(5.5).

$$
\begin{align*}
H_2 &= H_0^1(\Omega) \times L^2(\Omega) \times L_*(\Omega) \times L_*(\Omega), \\
D(A_2) &= (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_*^1(\Omega)) \\
&\quad \times (H^2(\Omega) \cap H_*^1(\Omega)),
\end{align*}
$$

where

$$
\begin{align*}
L_*(\Omega) &= \{y \in L^2(\Omega) | \int_0^1 y(x)dx = 0\}, \\
H_*^1(\Omega) &= \{y \in H^1(\Omega) | \int_0^1 y(x)dx = 0\}.
\end{align*}
$$
Using the same method as in Section 2, we can get the similar results as Lemma 2.1 for problems (5.1)-(5.9). Using these similar results as Lemma 2.1 and the semigroup theory, we can also prove the global existence and exponential stability theorems similar to Theorem 2.1 and Theorem 2.2. Next we can use the same methods as in Section 3 and Section 4 to prove the theorems on nonhomogeneous and semilinear thermodiffusion equations. We state these main theorems without proofs.
Theorem 5.1 (Homogeneous Problems)

Suppose that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), \( u_1 \in H^1_0(\Omega) \), \( \theta_{10} \in H^2(\Omega) \cap H^1(\Omega) \), \( \theta_{20} \in H^2(\Omega) \cap H^1_*(\Omega) \). Then the problem (5.1)-(5.5) has a unique classical solution \((u(t), u_t(t), \theta_1(t), \theta_2(t))\) such that

\[
\begin{align*}
    u(t) &\in C^1([0, +\infty), H^1_0(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)), \\
    u_t(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^1_0(\Omega)), \\
    \theta_1(t) &\in C^1([0, +\infty), L^*_2(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_*(\Omega)), \\
    \theta_2(t) &\in C^1([0, +\infty), L^*_2(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_*(\Omega)).
\end{align*}
\]

Furthermore, the total energy \( E(t) \) defined in (1.11) decays exponentially as time tends to infinity.
Theorem 5.2 (Nonhomogeneous Problems)

Suppose that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $u_1 \in H^1_0(\Omega)$, $\theta_{10} \in H^2(\Omega) \cap H^1_1(\Omega)$, $\theta_{20} \in H^2(\Omega) \cap H^1_1(\Omega)$ and $f(x, \cdot), g(x, \cdot), h(x, \cdot) \in C^1([0, +\infty), L^2(\Omega))$. Then the problem (5.1)-(5.5) has a unique classical solution $(u(t), u_t(t), \theta_1(t), \theta_2(t))$ such that

$$u(t) \in C^1([0, +\infty), H^1_0(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)),$$

$$u_t(t) \in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^1_0(\Omega)),$$

$$\theta_1(t) \in C^1([0, +\infty), L^2_*(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_1(\Omega)),$$

$$\theta_2(t) \in C^1([0, +\infty), L^2_*(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_1(\Omega)).$$
Theorem 5.2-Continuation

Furthermore, if \( \int_0^{+\infty} (\| f \|^2 + \| g \|^2 + \| h \|^2) dt < +\infty \), the total energy \( E(t) \) defined in (1.11) decays to zero as time tends to infinity. i.e., \( E(t) \) satisfies (3.17). If further, (1) \( f \), \( g \) and \( h \) satisfies (3.18), then the energy \( E(t) \) satisfies (3.19); (2) \( f \), \( g \) and \( h \) satisfies (3.20), then the energy \( E(t) \) satisfies (3.21).
Theorem 5.3 (Semilinear Problems)

Suppose that $f(z_1, z_2, z_3, z_4), g(z_1, z_2, z_3, z_4), h(z_1, z_2, z_3, z_4) \in C^1(R^4, R)$ and $\nabla f, \nabla g, \nabla h$ are uniformly bounded. Then for any $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega), \theta_{10} \in L^2_*(\Omega), \theta_{20} \in L^2_*(\Omega)$, the problem (5.1)-(5.5) has a unique mild solution $(u(t), u_t(t), \theta_1(t), \theta_2(t))$ such that

\[
\begin{align*}
    u(t) &\in C^1([0, +\infty), H_0^1(\Omega)), \\
    u_t(t) &\in C^1([0, +\infty), L^2(\Omega)), \\
    \theta_1(t) &\in C^1([0, +\infty), L^2_*(\Omega)), \\
    \theta_2(t) &\in C^1([0, +\infty), L^2_*(\Omega)).
\end{align*}
\]
Theorem 5.3-Continuation

Furthermore, if \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), \( u_1 \in H^1_0(\Omega) \), \( \theta_{10} \in H^2(\Omega) \cap H^1_0(\Omega) \), \( \theta_{20} \in H^2(\Omega) \cap H^1_*(\Omega) \), then the initial boundary value problem for the semilinear thermodiffusion equations (4.1)-(4.3), (1.4) and (1.6) has a unique classical solution \((u(t), u_t(t), \theta_1(t), \theta_2(t))\) such that

\[
\begin{align*}
  u(t) &\in C^1([0, +\infty), H^1_0(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)), \\
  u_t(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^1_0(\Omega)), \\
  \theta_1(t) &\in C^1([0, +\infty), L^2_*(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_*(\Omega)), \\
  \theta_2(t) &\in C^1([0, +\infty), L^2_*(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H^1_*(\Omega)).
\end{align*}
\]
Finally, we give the proper state spaces and the domains of operator $A_i$ $(i = 3, 4, 5)$ for problem (1.1)-(1.3), (1.4) subject to the boundary condition (1.7)-(1.9) respectively.

\[
\begin{align*}
H_3 &= H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2_*(\Omega), \\
D(A_3) &= (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \\
&\quad \times (H^2(\Omega) \cap H^1_*(\Omega)),
\end{align*}
\]

\[
\begin{align*}
H_4 &= H_0^1(\Omega) \times L^2(\Omega) \times L^2_*(\Omega) \times L^2(\Omega), \\
D(A_4) &= (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \\
&\quad \times (H^2(\Omega) \cap H_0^1(\Omega)),
\end{align*}
\]
\[
\begin{align*}
H_5 &= H^1_*(\Omega) \times L^2_*(\Omega) \times L^2(\Omega) \times L^2(\Omega), \\
D(A_5) &= \left\{ y \in H_5 \middle| u_x \in H^1_0(\Omega), u_t \in H^1_*(\Omega), \right. \\
&\quad \left. \theta_1, \theta_2 \in (H^2(\Omega) \cap H^1_0(\Omega)) \right\}.
\end{align*}
\]

Using the same methods as above, we can also get the same main theorem.