Multiple-point hit distribution functions and vague convergence of related measures

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Version: December 17, 2008

Abstract

For a stationary and isotropic random closed set $Z$ in $\mathbb{R}^d$ it is a well-known fact that its covariance $C(t)$ and its spherical contact distribution function $\tilde{H}_B(t)$ admit at $t = 0$ a derivative which is a multiple of the surface intensity of $Z$. Within the quite general setting of gentle sets, Kiderlen and Rataj [10] show a more general result (covering both previous cases) for the derivative of a hit distribution function of $Z$ with respect to a structuring element which only needs to be compact and should contain the origin. Using this general setting we introduce $m$-point hit distribution functions of $Z$, $m \geq 2$, and show how they are related to the $m$th-order surface product density of $Z$. This also generalizes a result of Ballani [1] for the two-point spherical contact distribution function of a germ-grain model.

Keywords: random closed set, gentle set, covariance, contact distribution, hit distribution, rose of directions, surface intensity, surface product density

2000 Mathematics Subject Classification: 60D05, 60G57

1 Introduction

For the description and analysis of random closed sets $Z$ in $\mathbb{R}^d$ ($d \geq 2$) different characteristics are fundamental, and it turns out that these characteristics are often interrelated in a surprising way.

It is a well-known fact [13, p. 204] that for a stationary and isotropic random closed set $Z$ the covariance $C(t) = \mathbb{P}(0 \in Z, q \in Z)$, $t = \|q\|$, $q \in \mathbb{R}^d$, admits at $t = 0$ a derivative

$$C'(0) = -\frac{b_{d-1}}{db_d} S_d^{(d)},$$

(1)

where $S_d^{(d)}$ is the surface intensity [13, p. 76] of $Z$ and $b_d$ is the volume of the Euclidean unit ball $B^d$ in $\mathbb{R}^d$. Since $C(t)$ can often not be expressed analytically, relationship (1) serves as the starting point for an exponential approximation of $C(t)$, see [13, 3].

In physics of porous media relationship (1) plays an important role since it is considered as a geometric constraint that must be obeyed by any physically realizable covariance [14, 5]. A heuristic explanation of (1), i.e. for the stationary and isotropic case, was already given by Delbye et al. [4] in 1957. Berryman [2, Equation (7)] generalized (1) to the stationary and anisotropic case by replacing $C(t)$ with the angular average $\int C(tw) \sigma_{d-1}/(db_d)$, where $\sigma_{d-1}$ is the spherical Lebesgue measure. So far, due to averaging over all directions no directional information was taken into account. Only recently, Kiderlen and Jensen [9, Theorem 4] in dimension $d = 2$, and Gokhale et al. [5, Equation (20)] gave an according result which (in principle) relates for any unit vector $w$ the directional derivative of $C(tw)$ at $t = 0$ to the surface intensity $S_d^{(d)}$. More precisely, they state that

$$\frac{\partial}{\partial t} \bigg|_{t=0} \mathbb{P}(0 \notin Z, tw \in Z) = \frac{S_d^{(d)}}{2} \int_{S_{d-1}} |\langle w, n \rangle| \mathcal{R}(dn)$$

(2)

for the case that the rose of directions of outer normal vectors $\mathcal{R}$ of $Z$ is even. Here, $S_{d-1}$ denotes the unit sphere in $\mathbb{R}^d$, and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^d$.

The contact distribution functions [13, 6, 11, 7] $\tilde{H}_B(t) = \mathbb{P}(o \in Z \oplus tB \mid o \notin Z)$, $t \geq 0$, of a stationary random closed set $Z$ with respect to the convex compact structuring element

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$B \subset \mathbb{R}^d$, having the origin in its interior, also admit at $t = 0$ a (right-hand) derivative. For example, for the spherical contact distribution function $\tilde{H}_s = \tilde{H}_{B^d}$ it is well known [13, Equation (6.2.4)] that

$$(1 - V_V^{(d)}) \tilde{H}_s'(0) = S_V^{(d)},$$

where $V_V^{(d)} = \mathbb{P}(o \in Z)$ is the volume fraction of $Z$. Here, $\oplus$ denotes Minkowski addition.

Up to the sign the similarity of (1) and (3) might be surprising for the first moment, nevertheless, as Kiderlen and Rataj [10] show, both relationships can be integrated in a more universally valid relationship. They do this for random closed sets which are almost surely gentle sets, which is a quite general class of sets containing, e.g., the extended convex ring. For a compact set $B \subset \mathbb{R}^d$ with $o \in B$ they introduce the hit distribution function $H_B(x,t) = \mathbb{P}(x \in Z \oplus tB \mid x \notin Z)$, which coincides with the contact distribution function $\tilde{H}_B$ if $B$ is star-shaped with respect to the origin. If $Z$ is stationary they show, writing then $H_B(t) = H_B(a,t)$, that

$$(1 - V_V^{(d)}) \frac{\partial}{\partial t} \bigg|_{t = 0^+} H_B(t) = S_V^{(d)} \int_{S^{d-1}} h(\hat{B}, u)\mathcal{R}(du),$$

where $h(B, \cdot)$ is the support function of the convex hull of $B$, and $\hat{B} = -B$. Inserting $B = \{o, w\}$, $w \in S^{d-1}$, in (4) yields (2) since

$$\mathbb{P}(0 \notin Z, tw \in Z) = \mathbb{P}(o \notin Z)\mathbb{P}(o \in Z \oplus t[-w, o] \mid o \notin Z).$$

In [1] Ballani introduces the two-point spherical contact distribution function (there denoted as second-order spherical contact distribution function) at $(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ by

$$H_s^{(2)}(x_1, t_1, x_2, t_2) = \mathbb{P}(x_1 \in Z \oplus t_1 B^d, x_2 \in Z \oplus t_2 B^d \mid x_1 \notin Z, x_2 \notin Z)$$

and shows that under certain conditions $H_s^{(2)}(x_1, t_1, x_2, t_2)$ admits a second partial right-hand derivative at $(t_1, t_2) = (0, 0)$ which is related to the second-order surface product density $\varrho_S^{(2)}(x_1, x_2)$ of $Z$ via

$$\mathbb{P}(x_1 \notin Z, x_2 \notin Z) \frac{\partial}{\partial t_2} \bigg|_{t_2 = 0^+} \frac{\partial}{\partial t_1} \bigg|_{t_1 = 0^+} H_s^{(2)}(x_1, t_1, x_2, t_2) = \varrho_S^{(2)}(x_1, x_2).$$

The second-order surface product density can be seen as the second-order analogue of the surface intensity and is defined as the Radon-Nikodým derivative (if it exists) of the second-order moment measure of the random surface measure $\mathcal{H}^{d-1}(\cdot \cap \partial Z)$ of $Z$, where $\mathcal{H}^k$ is the $k$-dimensional Hausdorff measure.

Using the approach and the setting of Kiderlen and Rataj [10] we show that at least the weaker relationship

$$(t_1, t_2)^{-1}\mathbb{P}(x_1 \notin Z, x_2 \notin Z) H_s^{(2)}_{B_1, B_2}(x_1, t_1, x_2, t_2) \mathcal{H}^{2d}(d(x_1, x_2))$$

$$\rightharpoonup \varrho_S^{(2)}(x_1, x_2) \int \mathcal{H}^{2d}(d(x_1, x_2))$$

as $(t_1, t_2) \to (0^+, 0^+)$ holds for compact $B_1, B_2 \subset \mathbb{R}^d$ with $o \in B_1, B_2$, where $\rightharpoonup$ denotes the vague convergence of measures and $\mathcal{R}^{(2)}(x_1, x_2, \cdot)$ (as the second-order analogue of $\mathcal{R}$) is the joint conditional distribution of the outer normals at $x_1$ and $x_2$ under the condition that $x_1, x_2 \in \partial Z$. In particular, we obtain for $w_1, w_2 \in S^{d-1}$

$$(t_1, t_2)^{-1}\mathbb{P}(x_1 \notin Z, x_1 + t_1 w_1 \notin Z, x_2 \notin Z, x_2 + t_2 w_2 \notin Z) \mathcal{H}^{2d}(d(x_1, x_2))$$

$$\rightharpoonup \varrho_S^{(2)}(x_1, x_2) \int \mathcal{H}^{2d}(d(x_1, x_2))$$

where $a^+ = \max\{0, a\}$ is the positive part of $a \in \mathbb{R}$. 2
2 Definitions

We use the same setting and notation as in [10]. In order to make our paper more readable we repeat the most important definitions.

The exoskeleton \(\text{exo}(A)\) of a closed set \(A \subseteq \mathbb{R}^d\) is the set of all \(z \in \mathbb{R}^d \setminus A\) which do not have a unique nearest point in \(A\) (with respect to Euclidean distance). The metric projection \(\xi_A : \mathbb{R}^d \setminus \text{exo}(A) \to A\) maps \(a \in \mathbb{R}^d \setminus \text{exo}(A)\) to its unique nearest point \(\xi_A(a)\) in \(A\). The set

\[
N(A) := \left\{ \left( \xi_A(z), \frac{z - \xi_A(z)}{\|z - \xi_A(z)\|} \right) : z \notin A \cup \text{exo}(A) \right\}
\]

is called the normal bundle of \(A\). \(\| \cdot \|\) denotes Euclidean norm. \(N(A)\) is a measurable subset of \(\partial A \times \mathbb{S}^{d-1}\). The reach function \(\delta(A; \cdot) : \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0, \infty)\) of \(A\) is defined by

\[
\delta(A; a, n) := \inf \{ t \geq 0 : a + tn \in \text{exo}(A) \}.
\]

\(\delta(A; \cdot)\) is positive on \(N(A)\). Furthermore, \(C_{d-1}(A; \cdot)\) is the image measure of \(\mathcal{H}^{d-1}\) on \(\partial A\) under the mapping \(a \mapsto (a, n(A; a))\), where \(n(A; a)\) is the \(\mathcal{H}^{d-1}\)-almost everywhere defined unique outer normal vector at \(a\) satisfying \((a, n(A; a)) \in N(A)\).

A closed set \(A \subseteq \mathbb{R}^d\) is said to be gentle [10] if, for all bounded Borel sets \(B \subseteq \mathbb{R}^d\), \(\mathcal{H}^{d-1}(N(\partial A) \cap (B \times \mathbb{S}^{d-1})) < \infty\), and, for \(\mathcal{H}^{d-1}\)-almost all \(a \in \partial A\), there are non-degenerate balls \(B_i\) and \(B_a\) containing both \(a\) with \(B_i \subseteq A\) and \(\text{int} B_a \subseteq \mathbb{R}^d \setminus A\).

3 Dilatation- and erosion-volumes

Proposition 3.1 ([10]). If \(A\) is a closed gentle set then there are uniquely determined signed measures \(\mu_0(\partial A; \cdot), \ldots, \mu_{d-1}(\partial A; \cdot)\) on \(\mathbb{R}^d \times \mathbb{S}^{d-1}\), vanishing outside \(N(\partial A)\), with the following property:

For any measurable bounded function \(f\) on \(\mathbb{R}^d\) with compact support, we have

\[
\int_{\mathbb{R}^d} f(z) \mathcal{H}^d(dz) = \sum_{i=1}^d i b_i \int_{N(\partial A)} \int_0^\infty t^{i-1} f(a + tn) \, dt \, \mu_{d-1}(\partial A; d(a, n)).
\]  

(8)

In particular we have

\[
2 \mu_{d-1}(\partial A; \cdot) = C_{d-1}(A; \cdot) + C_{d-1}^*(A; \cdot),
\]  

(9)

where \(C_{d-1}^*(A; \cdot)\) is the image measure of \(C_{d-1}(A; \cdot)\) under the reflection \((a, n) \mapsto (a, -n)\).

Given a compact subset \(M\) of \(\mathbb{R}^d\), denote by \(\tilde{M} = \{-x : x \in M\}\) the reflection of \(M\) and let

\[
h(M, u) = h(\text{conv} M, u) = \sup \{ \langle y, u \rangle : y \in M \}
\]

be the support function of \((\text{the convex hull of})\ M\).

The following theorem is a straightforward generalization of Theorem 1 in [10] to our situation. Though in the remainder of this paper we will only use some special cases we state this theorem for the same general structuring as in [10, Theorem 1].

Theorem 3.1. Let \(A\) be a closed gentle set, \(m \geq 1\) an integer, \(C \subset (\mathbb{R}^d)^m\) a bounded Borel set and \(B_k, W_k\) and \(P_k, Q_k\), \(k = 1, \ldots, m\), \(4m\) non-empty compact subsets of \(\mathbb{R}^d\). For \(\varepsilon > 0\) set

\[
g_{k, \varepsilon}(z) = 1_{\{z + \varepsilon B_k \subseteq A \oplus \varepsilon P_k\}} 1_{\{z + \varepsilon W_k \cup (A \ominus \varepsilon Q_k)\}}, \quad k = 1, \ldots, m.
\]

Then

\[
\lim_{\varepsilon_1, \ldots, \varepsilon_m \to 0} \prod_{k=1}^m \varepsilon_k^{-1} \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \mathbf{1}_C(z_1, \ldots, z_m) \prod_{k=1}^m g_{k, \varepsilon}(z_k) \mathcal{H}^d(dz_1) \cdots \mathcal{H}^d(dz_m)
\]

\[
= 2^m \int_{N(A)} \ldots \int_{N(A)} \mathbf{1}_C(a_1, \ldots, a_m) \prod_{k=1}^m \left( h(P_k + \hat{Q}_k, n_k) - h(B_k \oplus W_k, n) \right)^+ \times C_{d-1}(A; d(a_1, n_1)) \cdots C_{d-1}(A; d(a_m, n_m)).
\]  

(10)
Proof. The proof uses ideas from the proofs of Theorem 1 in [10] and Theorem 3.1 in [1]. Since the case $m = 1$ was proved in [10, Theorem 1] we assume $m \geq 2$ in what follows. It can be assumed without loss of generality that $B_k \cup W_k \cup P_k \cup Q_k$ is contained in the ball $B(o,1/2)$ for all $k \in \{1, \ldots, m\}$. We iteratively apply Proposition 3.1, first to the function $f_1(z_1, \ldots, z_m)$, 

$$f_1(z_1, \ldots, z_m) = \begin{cases} 1_C(\xi \partial A(z_1), \ldots, \xi \partial A(z_m)) g_{1; \varepsilon_1}(z_1), & z_1 \notin \text{exo}(\partial A), \\ 0, & z_1 \in \text{exo}(\partial A), \end{cases}$$

giving

$$\int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} 1_C(\xi \partial A(z_1), \ldots, \xi \partial A(z_m)) \prod_{k=1}^m g_{k; \varepsilon_k}(z_k) \mathcal{H}^d(dz_1) \cdots \mathcal{H}^d(dz_m)$$

$$= \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} f(z_1, \ldots, z_m) \mathcal{H}^d(dz_1) \cdots \mathcal{H}^d(dz_m)$$

$$= \sum_{j_1=1}^m j_1 b_1 \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \delta(\partial A; a_1, n_1) \prod_{k=2}^m \mu_{d-j_1}(\partial A; d(a_1, n_1)) \mathcal{H}^d(dz_2) \cdots \mathcal{H}^d(dz_m)$$

$$= \sum_{j_1=1}^m j_1 b_1 \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \delta(\partial A; a_1, n_1) \prod_{k=2}^m \mu_{d-j_1}(\partial A; d(a_1, n_1))$$

$$\times g_{k; \varepsilon_k}(a_1 + t_1 n_1) \prod_{k=2}^m g_{k; \varepsilon_k}(z_k) \mathcal{H}^d(dz_2) \cdots \mathcal{H}^d(dz_m)$$

and then for $k = 2, \ldots, m$ to $f_k(\ldots, z_k, \ldots)$,

$$f_k(a_1, \ldots, a_{k-1}, z_k, \ldots, z_m) = \begin{cases} 1_C(a_1, \ldots, a_{k-1}, \xi \partial A(z_k), \ldots, \xi \partial A(z_m)) g_{k; \varepsilon_k}(z_k), & z_k \notin \text{exo}(\partial A), \\ 0, & z_k \in \text{exo}(\partial A), \end{cases}$$

finally giving

$$\int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} 1_C(\xi \partial A(z_1), \ldots, \xi \partial A(z_m)) \prod_{k=1}^m g_{k; \varepsilon_k}(z_k) \mathcal{H}^d(dz_1) \cdots \mathcal{H}^d(dz_m)$$

$$= \sum_{j_1=1}^m j_1 b_1 \ldots \sum_{j_m=1}^m j_m b_m \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \delta(\partial A; a_1, n_1) \cdots \delta(\partial A; a_m, n_m) \prod_{k=1}^m \mu_{d-j_1}(\partial A; d(a_1, n_1)) \cdots \mu_{d-j_m}(\partial A; d(a_m, n_m))$$

$$\times \mu_{d-j_1}(\partial A; d(a_1, n_1)) \cdots \mu_{d-j_m}(\partial A; d(a_m, n_m))$$

where we have also tacitly used Fubini’s theorem (which can be applied because of (2.25) in [8]). For each $k \in \{1, \ldots, m\}$ we have that $|g_{k; \varepsilon_k}|$ is bounded by 1 and since $B_k \neq \emptyset$ and $P_k$ are contained in $B(o,1/2)$, the support of $g_{k; \varepsilon_k}$ is contained in $A \oplus B(o,\varepsilon_k)$. Furthermore, since $C \subset (\mathbb{R}^d)^m$ is bounded, there are bounded Borel sets $C_1, \ldots, C_m \subset \mathbb{R}^d$ such that
$C \subseteq C_1 \times \ldots \times C_m$. Hence we have

$$\left| \int_{N(\partial A)} \delta(\partial A, a_1, n_1) \cdots \int_{N(\partial A)} \delta(\partial A, a_m, n_m) \sum_{k=1}^{m} \frac{\mu_{d-j_k}(\partial A; d(a_k + t_k n_k))}{\varepsilon_k} \cdots \mu_{d-j_m}(\partial A; d(a_m, n_m)) \right|$$

$$\leq \prod_{k=1}^{m} \varepsilon_k^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{N(\partial A)} \int_{N(\partial A)} 1_{C_1}(a_1, \ldots, a_m) \prod_{k=1}^{m} \frac{\mu_{d-j_k}(\partial A; d(a_k + t_k n_k))}{\varepsilon_k} \cdots \mu_{d-j_m}(\partial A; d(a_m, n_m)) \cdots \mu_{d-j_m}(\partial A; d(a_m, n_m)) \cdots \|\mu_{d-j_m}(\partial A; d(a_m, n_m))\| dt_1 \cdots dt_m$$

and the total variation measure $|\mu_{d-j_k}(\partial A; C_k \times S^{d-1})|$ is finite for any $j_k$ since $C_k$ is bounded. Thus the last expression tends to $0$ as $(\varepsilon_1, \ldots, \varepsilon_m) \to (0^+, \ldots, 0^+)$ whenever there is a $k \in \{1, \ldots, m\}$ with $j_k > 1$. This implies

$$\lim_{\varepsilon_1, \ldots, \varepsilon_m \to 0^+} \prod_{k=1}^{m} \varepsilon_k^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{N(\partial A)} \int_{N(\partial A)} 1_{C_1}(z_1, \ldots, z_m) \prod_{k=1}^{m} g_{k;\varepsilon}(z_k) |\mathcal{H}^d(dz_1) \cdots \mathcal{H}^d(dz_m)|$$

$$= 2^m \lim_{\varepsilon_1, \ldots, \varepsilon_m \to 0^+} \int_{N(A)} \int_{N(A)} \int_{N(A)} 1_{C}(a_1, \ldots, a_m) \prod_{k=1}^{m} \frac{g_{k;\varepsilon}(a_k + t_k n_k)}{\varepsilon_k} dt_k$$

$$\times \mu_{d-j_1}(\partial A; d(a_1, n_1)) \cdots \mu_{d-j_m}(\partial A; d(a_1, n_1))$$

$$= 2^m \lim_{\varepsilon_1, \ldots, \varepsilon_m \to 0^+} \int_{N(A)} \int_{N(A)} \int_{N(A)} 1_{C}(a_1, \ldots, a_m) \prod_{k=1}^{m} G_{k;\varepsilon}(a_k, n_k)$$

$$\times \mu_{d-j_1}(\partial A; d(a_1, n_1)) \cdots \mu_{d-j_m}(\partial A; d(a_1, n_1))$$

where we have used (9), $N(A) \subseteq N(\partial A)$ and the abbreviation

$$G_{k;\varepsilon}(a_k, n_k) = \int_{\delta(\partial A, a_k, n_k)} \frac{g_{k;\varepsilon}(a_k + t_k n_k)}{\varepsilon_k} dt_k, \quad k = 1, \ldots, m.$$ 

From [10, (11)] we have

$$G_{k;\varepsilon}(a_k, n_k) \to (h(P_k \oplus Q_k, n_k) - h(Q_k \oplus W_k, n_k))^+, \quad \varepsilon_k \to 0^+.$$ 

Since $G_{k;\varepsilon}(z_k) \leq 1_{(A \ominus B(z_k)) \cap (A^* \ominus B(z_k))}(z_k)$ for all $k \in \{1, \ldots, m\}$ implies

$$0 \leq 1_{C}(a_1, \ldots, a_m) \prod_{k=1}^{m} G_{k;\varepsilon}(a_k, n_k) \leq 1_{C}(a_1, \ldots, a_m) \prod_{k=1}^{m} \frac{1}{\varepsilon_k} \int_{\delta(\partial A, a_k, n_k)} 1_{\{x_k \leq t_k \leq \varepsilon_k\}} dt_k$$

$$\leq 2^m 1_{C}(a_1, \ldots, a_m)$$

which yields a uniformly integrable upper bound, the proof is completed by applying Lebesgue’s dominated convergence theorem.

\[ \Box \]

**Remark 3.1.** Note that the assertion of Theorem 3.1 using the limit $\lim_{\varepsilon_1, \ldots, \varepsilon_m \to 0^+}$ is stronger than using the iterated limit $\lim_{\varepsilon_1 \to 0^+} \ldots \lim_{\varepsilon_m \to 0^+}$. Clearly, using this iterated limit instead, an analogous assertion to Theorem 3.1 can be proved by simply applying Theorem 1 in [10] $m$ times.
Corollary 3.1. Let $A$ be a closed gentle set, $m \geq 1$ an integer, $C \subset (\mathbb{R}^d)^m$ a bounded Borel set. Fix non-empty compact subsets $B_k$, $k = 1, \ldots, m$, of $\mathbb{R}^d$. Then

$$
\lim_{\varepsilon_1, \ldots, \varepsilon_m \to 0^+} \prod_{k=1}^m \varepsilon_k^{-1} \int \cdots \int 1_C(z_1, \ldots, z_m) \prod_{k=1}^m 1_{[A \subset \varepsilon_k B_k] \setminus A}(z_k) \mathcal{H}^d(dz_1) \cdots \mathcal{H}^d(dz_m)
$$

$$= 2^m \int_{N(A)} \cdots \int_{N(A)} 1_C(a_1, \ldots, a_m) \prod_{k=1}^m h^+(B_k, n)
$$

$$\times C_{d-1}(A; d(a_1, u_1)) \cdots C_{d-1}(A; d(a_m, n_m))
$$

$$= 2^m \int_{N(A)} \cdots \int_{N(A)} 1_C(a_1, \ldots, a_m) \prod_{k=1}^m h(B_k \cup \{o\}, n)
$$

$$\times C_{d-1}(A; d(a_1, u_1)) \cdots C_{d-1}(A; d(a_m, n_m)),
$$

(11)

4 $m$-point hit distribution functions

We will need the following integrability condition: We require that

$$E \left[ \prod_{k=1}^m \mathcal{H}^{d-1}(N(\partial Z) \cap D_k \times S^{d-1}) \right] < \infty
$$

(12)

holds for all bounded Borel sets $D_k \subset \mathbb{R}^d$, $k = 1, \ldots, m$.

If (12) holds, then $\Lambda_{d-1}^{(m)}$, defined by

$$\Lambda_{d-1}^{(m)}(D_1 \times \cdots \times D_m) = E \left[ \prod_{k=1}^m C_{d-1}(Z; D_k) \right], \quad D_1, \ldots, D_m \subset \mathbb{R}^d \times S^{d-1} \text{ Borel},
$$

is a locally finite (nonnegative) measure, concentrated on $(\mathbb{R}^d \times S^{d-1})^m$. It is called the $m$th-order moment measure of $C_{d-1}(Z; \cdot)$. A disintegration of this measure yields

$$\Lambda_{d-1}^{(m)}(d(z_1, n_1, \ldots, z_m, n_m)) = \mathcal{R}^{(m)}(z_1, \ldots, z_m, d(n_1, \ldots, n_m)) \Lambda_{d-1}^{(m)}(dz_1 \times S^{d-1} \times \cdots \times dz_m \times S^{d-1}),
$$

(13)

where $\mathcal{R}^{(m)}$ is a stochastic kernel from $(\mathbb{R}^d)^m$ to $(S^{d-1})^m$. $\mathcal{R}^{(m)}$ can be interpreted as a joint conditional directional distribution in the sense of a $m$-point mark distribution [13, p. 114], where the marks are outer normal vectors from $S^{d-1}$.

Let

$$v^{(m)}(z_1, \ldots, z_m) = P(z_1 \notin Z, \ldots, z_m \notin Z)
$$

be the $m$-point void probability of $Z$ at $z_1, \ldots, z_m \in \mathbb{R}^d$, and fix arbitrary compact sets $B_1, \ldots, B_m \subset \mathbb{R}^d$ with $o \in B_k$, $k = 1, \ldots, m$. If $v^{(m)}(z_1, \ldots, z_m) > 0$, then the $m$-point hit distribution function at $(z_1, \ldots, z_m)$ with structuring elements $B_1, \ldots, B_m$ is defined by

$$H^{(m)}_{B_1, \ldots, B_m}(z_1, t_1, \ldots, z_m, t_m) = P(z_1 \in Z \oplus t_1 \bar{B}_1, \ldots, z_m \in Z \oplus t_m \bar{B}_m \mid z_1 \notin Z, \ldots, z_m \notin Z).
$$

For $v^{(m)}(z_1, \ldots, z_m) = 0$, we set $H^{(m)}_{B_1, \ldots, B_m}(z_1, t_1, \ldots, z_m, t_m) = 1$.

Theorem 4.1. Let $m \geq 1$ be an integer, and let $B_k, \subset \mathbb{R}^d$, $k = 1, \ldots, m$, be non-empty compact sets with $o \in B_1, \ldots, B_m$. Let $Z$ be a.s. a gentle set such that (12) holds for all bounded Borel sets $D_1, \ldots, D_m \subset \mathbb{R}^d$. Then

$$\prod_{k=1}^m t_k^{-1} v^{(m)}(z_1, \ldots, z_m) H^{(m)}_{B_1, \ldots, B_m}(z_1, t_1, \ldots, z_m, t_m) \mathcal{H}^{md}(d(z_1, \ldots, z_m))
$$

$$= \frac{2^m}{v^{(m)}} \int_{(S^{d-1})^m} \prod_{k=1}^m h(\bar{B}_k, n_k) \mathcal{R}^{(m)}(z_1, \ldots, z_m, d(n_1, \ldots, n_m))
$$

$$\times \Lambda_{d-1}^{(m)}(dz_1 \times S^{d-1} \times \cdots \times dz_m \times S^{d-1}),
$$

(14)

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as \((t_1, \ldots, t_m) \to (0^+, \ldots, 0^+),\) where \(\Rightarrow\) denotes the vague convergence of measures.

**Proof.** The proof uses ideas from the proof of [10, Theorem 4]. Without loss of generality, we may assume that \(B_1, \ldots, B_m \subseteq B(o, 1)\). If \(g : (\mathbb{R}^d)^m \to \mathbb{R}\) is a continuous function with compact support, we have

\[
\int_{(\mathbb{R}^d)^m} g(z_1, \ldots, z_m) \mathcal{H}^m(z_1, \ldots, z_m) \mathcal{H}^m(d(z_1, \ldots, z_m)) = \mathbb{E} \left[ \int_{(\mathbb{R}^d)^m} g(z_1, \ldots, z_m) \prod_{k=1}^m 1_{[Z \oplus t_k \cdot B_k] \setminus Z}(z_k) \mathcal{H}^m(d(z_1, \ldots, z_m)) \right]
\]

\[
= \mathbb{E} \left[ \int_{(\mathbb{R}^d)^m} (g(z_1, \ldots, z_m) - g(\xi_0 z_1, \ldots, \xi_0 z_m)) \prod_{k=1}^m 1_{[Z \oplus t_k \cdot B_k] \setminus Z}(z_k) \mathcal{H}^m(d(z_1, \ldots, z_m)) \right]
\]

\[
+ \mathbb{E} \left[ \int_{(\mathbb{R}^d)^m} g(\xi_0 z_1, \ldots, \xi_0 z_m) \prod_{k=1}^m 1_{[Z \oplus t_k \cdot B_k] \setminus Z}(z_k) \mathcal{H}^m(d(z_1, \ldots, z_m)) \right]
\]

\[
= R_1(t_1, \ldots, t_m) + R_2(t_1, \ldots, t_m).
\]

If \(z_k \in [Z \oplus t_k \cdot B_k] \setminus Z\), then \(z_k \in [Z \oplus B(o, t_k)] \setminus Z\), and thus \(|z_k - \xi_0 z_k| = t_k, k = 1, \ldots, m\), and \(|(z_1, \ldots, z_m) - (\xi_0 z_1, \ldots, \xi_0 z_m)| \leq \|(t_1, \ldots, t_m)\|\). Denote the support of \(g\) by \(\text{supp}\ g\).

Since \(g\) is compact there are compact sets \(D_1, \ldots, D_m \subseteq \mathbb{R}^d\) such that \(\text{supp}\ g \subseteq D_1 \times \ldots \times D_m\). Define the compact sets \(D_k = D_k \oplus B(o, 1), k = 1, \ldots, m\). As \(g\) is uniformly continuous, there is \(0 < t_\varepsilon \leq 1\) such that \(|g(z_1, \ldots, z_m) - g(\xi_0 z_1, \ldots, \xi_0 z_m)| \leq \varepsilon\) for all \((z_1, \ldots, z_m) \in ([Z \oplus t \cdot B_1] \setminus Z) \times \ldots \times ([Z \oplus t \cdot B_1] \setminus Z)\) whenever \(|(t_1, \ldots, t_m)| < t_\varepsilon\). For these \((t_1, \ldots, t_m)\), from Proposition 3.1 we have

\[
|R_1(t_1, \ldots, t_m)| \leq \varepsilon \mathbb{E} \left[ \prod_{k=1}^m \mathcal{H}^d(\xi_0^{-1}(D_k) \cap [Z \oplus t_k \cdot B_k] \setminus Z) \right]
\]

\[
\leq \varepsilon \mathbb{E} \left[ \prod_{k=1}^m \sum_{j_k=1}^d b_{j_k} \int_{N(\partial Z)} 1_{D_k}(a_k) \int_0^{t_k} s_{k}^{j_k-1} ds_k |\partial Z; d(a_k, n_k)| \right]
\]

\[
\leq \varepsilon \mathbb{E} \left[ \prod_{k=1}^m \sum_{j_k=1}^d b_{j_k} \int_{N(\partial Z)} 1_{D_k \times S^{d-1}}(\partial Z; d(a_k, n_k)) \right]
\]

Hence we have

\[
\frac{|R_1(t_1, \ldots, t_m)|}{t_1 \cdot \ldots \cdot t_m} \leq \varepsilon \mathbb{E} \left[ \prod_{k=1}^m \sum_{j_k=1}^d b_{j_k} \int_{N(\partial Z)} 1_{D_k \times S^{d-1}}(\partial Z; d(a_k, n_k)) \right].
\]

In view of [8, (2.13) and Corollary 2.5] and (12), the expectation on the right-hand side is finite, and since \(\varepsilon > 0\) was arbitrary, we obtain

\[
\frac{|R_1(t_1, \ldots, t_m)|}{t_1 \cdot \ldots \cdot t_m} \to 0
\]

as \((t_1, \ldots, t_m) \to (0^+, \ldots, 0^+)\). According to Corollary 3.1, the integral in \(R_2(t_1, \ldots, t_m)\) satisfies

\[
\prod_{k=1}^m t_k^{-1} \int_{(\mathbb{R}^d)^m} g(\xi_0 z_1, \ldots, \xi_0 z_m) \prod_{k=1}^m 1_{[Z \oplus t_k \cdot B_k] \setminus Z}(z_k) \mathcal{H}^m(d(z_1, \ldots, z_m))
\]

\[
= 2^m \int_{N(Z)} \ldots \int_{N(Z)} g(z_1, \ldots, z_m) \prod_{k=1}^m h(B_k, n_k) C_{d-1}(Z, d(z_1, n_1)) \ldots C_{d-1}(Z, d(z_m, n_m))
\]
as \((t_1, \ldots, t_m) \to (0^+, \ldots, 0^+)\). In view of the dominating terms in the proof of Theorem
3.1 and (12), Lebesgue’s dominated convergence theorem allows us to interchange limit and
expectation, and we get

\[
\lim_{t_1, \ldots, t_m \to 0^+} \prod_{k=1}^m t_k^{-1} \int_{(\mathbb{R}^d)^m} g(z_1, \ldots, z_m) v^{(m)}(z_1, \ldots, z_m) H^{(m)}_{t_1, \ldots, t_m}(z_1, t_1, \ldots, z_m, t_m) \\
\times \mathcal{H}^m(d(z_1, \ldots, z_m)) = 2^m \int_{(\mathbb{R}^d \times S^{d-1})^m} g(z_1, \ldots, z_m) \prod_{k=1}^m h(\tilde{B}_k, n_k) \Lambda_{d-1}^{(m)}(d(z_1, n_1, \ldots, z_m, n_m)).
\]

Then (13) completes the proof of the theorem. \(\square\)

**Corollary 4.1.** Let \(m \geq 1\) be an integer. Let \(Z\) be a.s. a gentle set such that (12) holds
for all bounded Borel sets \(D_1, \ldots, D_m \subseteq \mathbb{R}^d\). Assume that \(\Lambda^{(m)}_{d-1} \cdot \times S^{d-1} \times \cdots \times S^{d-1}\) is
absolutely continuous with respect to \(\mathcal{H}^m\) with density \(\lambda_{d-1}\).

(a) Then

\[
\left( \prod_{k=1}^m t_k^{-1} \right) v^{(m)}(z_1, \ldots, z_m) H^{(m)}(z_1, t_1, \ldots, z_m, t_m) \mathcal{H}^m(d(z_1, \ldots, z_m)) \\
\to 2^m \Lambda^{(m)}_{d-1}(z_1, \ldots, z_m) \mathcal{H}^m(d(z_1, \ldots, z_m))
\]

as \((t_1, \ldots, t_m) \to (0^+, \ldots, 0^+)\), where

\[
H^{(m)} := H^{(m)}_{B_d^c, \ldots, B_d}
\]

is the \(m\)-point spherical contact distribution function of \(Z\).

(b) Let \(w_1, \ldots, w_m \in S^{d-1}\). Then

\[
\left( \prod_{k=1}^m t_k^{-1} \right) v^{(m)}(z_1, \ldots, z_m) H^{(m)}_{w_1, \ldots, w_m}(z_1, t_1, \ldots, z_m, t_m) \mathcal{H}^m(d(z_1, \ldots, z_m)) \\
\to 2^m \Lambda^{(m)}_{d-1}(z_1, \ldots, z_m) \int_{(S^{d-1})^m} \prod_{k=1}^m \langle -w_k, n_k \rangle^+ \mathcal{R}^{(m)}(z_1, \ldots, z_m, d(n_1, \ldots, n_m)) \\
\times \mathcal{H}^m(d(z_1, \ldots, z_m))
\]

as \((t_1, \ldots, t_m) \to (0^+, \ldots, 0^+)\), where

\[
H^{(m)}_{w_1, \ldots, w_m} := H^{(m)}_{[0, w_1], \ldots, [0, w_m]}
\]

is the \(m\)-point linear contact distribution function of \(Z\) with respect to \(w_1, \ldots, w_m\).

(c) Let \(w_1, \ldots, w_m \in S^{d-1}\). Then

\[
\left( \prod_{k=1}^m t_k^{-1} \right) \tilde{C}^{(m)}_{w_1, \ldots, w_m}(z_1, t_1, \ldots, z_m, t_m) \mathcal{H}^m(d(z_1, \ldots, z_m)) \\
\to 2^m \Lambda^{(m)}_{d-1}(z_1, \ldots, z_m) \int_{(S^{d-1})^m} \prod_{k=1}^m \langle -w_k, n_k \rangle^+ \mathcal{R}^{(m)}(z_1, \ldots, z_m, d(n_1, \ldots, n_m)) \\
\times \mathcal{H}^m(d(z_1, \ldots, z_m))
\]

as \((t_1, \ldots, t_m) \to (0^+, \ldots, 0^+)\), where

\[
\tilde{C}^{(m)}_{w_1, \ldots, w_m}(z_1, t_1, \ldots, z_m, t_m) := v^{(m)}(z_1, \ldots, z_m) H^{(m)}_{[0, w_1], \ldots, [0, w_m]}(z_1, t_1, \ldots, z_m, t_m) \\
= P(z_1 \notin Z, z_1 + t_1 w_1 \in Z, \ldots, z_m \notin Z, z_m + t_m w_m \in Z).
\]

In case \(m = 2\) and \(Z\) is a germ-grain model with convex grains, Corollary 4.1(a) coincides
with Theorem 3.1 in [1].
References


