19th C5 Graph Theory Workshop
"Cycles, Colourings, Cliques, Claws and Closures"

Kurort Rathen, May 04-08, 2015
SELECTED PROBLEMS

http://www.mathe.tu-freiberg.de/inst/theomath/?seite=rathen
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Colourings

1 \textbf{b-COLOURINGS OF GRAPHS}

(Mais Alkhateeb, Anja Kohl)

Let $G$ be a simple and undirected graph of order $n$. A $b$-coloring of $G$ is a proper vertex coloring such that there is a vertex in each color class, which is adjacent to at least one vertex in every other color class. Such a vertex is called a color-dominating vertex. The $b$-chromatic number of a graph $G$, denoted by $\chi_b(G)$, is the largest $k$ such that there is a $b$-coloring of $G$ by $k$ colors.

An interesting problem is to characterize the $d$-regular graphs $G$ satisfying $\chi_b(G) = d + 1$. This equality is e.g. satisfied for every $d$-regular graph with at least $2d^3$ vertices (Cabello and Jakovac ([3])) and for every $d$-regular graph with girth $g \geq 6$ (independently proved by El Sahili et al., Kouider et al., and Blidia et al.).

\textbf{Conjecture 1.1.} (Blidia, Maffray, and Zemir ([2])) Every $d$-regular graph $G$ with girth $g \geq 5$ which is not the Petersen graph satisfies $\chi_b(G) = d + 1$.

A graph $G$ is called b-continuous if it has a $b$-coloring by $k$ colors for all integers $k$ satisfying $\chi(G) \leq k \leq \chi_b(G)$. There exist graphs which are not b-continuous, e.g. every $(r - 1)$-factor of the complete bipartite graph $K_{r,r}$ for $r \geq 4$.

Recently we could verify that graphs $G$ with minimum degree $\delta(G) \geq n - 3$ are b-continuous ([1]). Moreover, there exist non-b-continuous graphs $G$ of order $n$ and minimum degree $\delta(G) = n - 5$. So we ask:

\textbf{Question 1.1.} Is every graph $G$ with minimum degree $\delta(G) = n - 4$ b-continuous?

Since all non-b-continuous graphs that are known so far contain a claw as an induced subgraph, we ask:

\textbf{Question 1.2.} Does there exist a claw-free graph that is not b-continuous?

We conjecture that there is no such graph. This is reason to pose the following conjecture:

\textbf{Conjecture 1.2.} Line graphs are b-continuous.

So far we proved it for line graphs of 2-degenerate graphs.
2 On the Cyclic Chromatic Number of 3-connected Plane Graphs

(Mirko Horňák)

The cyclic chromatic number of a plane graph $G$, in symbol $\chi_c(G)$, is a minimum number of colours in such a vertex colouring of $G$ that distinct vertices incident with a common face receive distinct colours. If $G$ is 2-connected, then $\chi_c(G) \geq \Delta^*(G)$, where $\Delta^*(G)$ is the maximum face degree of $G$. On the other hand, no 3-connected plane graph $G$ is known with $\chi_c(G) > \Delta^*(G) + 2$. Plummer and Toft ([7]) proved that $\chi_c(G) \leq \Delta^*(G) + 9$ and conjectured (PTC) that $\chi_c(G) \leq \Delta^*(G) + 2$ for any 3-connected plane graph $G$. Let PTC($d$) denote PTC restricted to 3-connected plane graphs $G$ with $\Delta^*(G) = d$. It is known that PTC($d$) is true for $d = 3$ (Four Colour Theorem), $d = 4$ (Borodin [1]), $d \in \{18, \ldots, 23\}$ (Horňák and Zlámalová [6]) and $d \geq 24$ (Horňák and Jendrol’ [5]). For $\Delta^*(G) \geq 60$ Enomoto et al. ([4]) obtained the best possible inequality: $\chi_c(G) \leq \Delta^*(G) + 1$ (graphs of pyramids show that the bound $\Delta^*(G) + 1$ cannot be improved).

The best general upper bound known so far is due to Enomoto and Horňák ([3]), namely $\chi_c(G) \leq \Delta^*(G) + 5$.

Problem 2.1. Prove PTC($d$) for some $d \in \{5, \ldots, 17\}$.

References:


3 Packing colouring in some special classes of planar graphs

(Jan Ekstein, Přemysl Holub)

Let $G$ be a connected graph and let $\text{dist}_G(u, v)$ denote the distance between vertices $u$ and $v$ in $G$. A partition of the vertex set of $G$ into (not necessarily nonempty) disjoint classes $X_1, ..., X_k$ such that each colour class $X_i$ is an $i$-packing is called a packing $k$-colouring. Each $i$-packing $X_i$ is a set of vertices such that any distinct pair $u, v \in X_i$ satisfies $\text{dist}_G(u, v) > i$. The smallest integer $k$ for which there exists a packing $k$-colouring of $G$ is called the packing chromatic number of $G$, denoted $\chi_\rho(G)$.

The determination of the packing chromatic number is known to be $\mathcal{NP}$-hard for general graphs [3] and Fiala and Golovach [1] showed that the problem remains $\mathcal{NP}$-hard even for trees. Sloper proved in [4] that the packing chromatic number of a complete infinite binary tree is equal to 7 while a complete infinite ternary tree cannot be coloured using a finite number of colours. Hence we will focus on cubic graphs. Since there are planar graphs for which $\chi_\rho$ is not finite (e.g. the infinite triangular grid [3]), we can consider some special subclasses of planar graphs. Thus we formulate the following questions:

**Problem 3.1.** Is there a constant $c$ such that, if $G$ is a cubic planar graph, then $\chi_\rho(G) \leq c$?

**Problem 3.2.** Is there a constant $c$ such that, if $G$ is a outerplanar graph, then $\chi_\rho(G) \leq c$?

References:


4 COLOURING VERTICES OF PLANE GRAPHS UNDER
RESTRICTIONS GIVEN BY FACES

(Stanislav Jendrol’)

Consider a vertex colouring of a connected plane graph G. A colour c is used k times by a face α of G if it appears k times along the facial walk of α. Two natural problems arise.

1. A vertex colouring ϕ is a weak parity vertex colouring of a connected plane graph G with respect to its faces if each face of G uses at least one colour an odd number of times. Problem is to determine the minimum number $\chi_w(G)$ of colours used in a wpv colouring of G.

In [1] it is proved that $\chi_w(G) \leq 4$ for every connected plane graph G with minimum degree at least 3. We strongly believe that the following holds.

Conjecture 4.1. Let G be a connected plane graph of minimum face degree at least 3. Then

$$\chi_w(G) \leq 3.$$  

The Conjecture is true for 2-connected cubic plane graphs, see [1].

2. A vertex colouring ϕ is a strong parity vertex colouring of a 2-connected plane graph G with respect to the faces of G if each face of G that uses a colour then it uses an odd number of times. Problem is to find the minimum number $\chi_s(G)$ of colours used in an spv colouring of G. We believe that

Conjecture 4.2. There is a constant k such that for every 2-connected plane graph G

$$\chi_s(G) \leq k.$$  

We do not know any 2-connected plane graph H with $\chi_s(H) \geq 7$. Hence, we believe that $k = 6$ in the above conjecture.
5 List colourings of integer distance graphs

(Arnfried Kemnitz)

Let $D$ be a subset of the positive integers $\mathbb{N}$. The integer distance graph $G(\mathbb{Z} , D) = G(D)$ is defined as the graph with the set of integers as vertex set, $V(G(D)) = \mathbb{Z}$, and edge set consisting of all pairs $uv$ whose distance $|u - v|$ is an element of the so-called distance set $D$.

General bounds for the chromatic number of integer distance graphs are

$$2 \leq \chi(G(D)) \leq |D| + 1.$$ 

Voigt ([4]) and Zhu ([5]) determined $\chi(G(D))$ if $|D| = 3$:
If $D = \{x , y , z\}$ consists of integers whose greatest common divisor equals 1, then $\chi(D) = 4$ if and only if $D = \{1, 2, 3n\}$ or $D = \{x , y , x + y\}$ and $x \neq y \pmod{3}$. If $x, y, z$ are odd then $\chi(D) = 2$. For all other 3-element distance sets $D$ it holds $\chi(D) = 3$.

General bounds for the list chromatic number (choice number) of integer distance graphs are $\chi(D) \leq \text{ch}(D) \leq |D| + 1$ (Kemnitz, Marangio 2001).

Question 5.1. Does there exist a 3-element distance set such that $\text{ch}(D) < 4$?

References:


Discussiones Mathematicae Graph Theory 22 (2002), 149-158.

Ars Combinatoria 52 (1999), 3-12.

[5] X. Zhu, Distance graphs on the real line
manuscript, 1996.
6 IMPROPER COLOURING OF PLANAR GRAPHS WITHOUT 4- AND 5-CYCLES

(André Raspaud)

**Definition 6.1.** A simple graphs \( G = (V, E) \) is called \((d_1, d_2, \ldots, d_k)\)-colourable if there is a partition \( V = V_1 \cup V_2 \cup \cdots \cup V_k \) of the vertex set such that the maximum degree of \( G_i = [V_i] \) is at most \( d_i \) for \( i = 1, \ldots, k \).

**Conjecture 6.1** (Steinberg, 1976). Every planar graph without 4- and 5-cycles is 3-colourable ((0,0,0)-colourable)

**Question 6.1.** Let \( G \) be a planar graph without 4- and 5-cycles. Can we prove \((1, 0, 0)\)- or \((1, 1, 0)\)- or \((2, 0, 0)\)- or \( \ldots \) -colourability of \( G \)?

**Remark:** \((7, 0, 0)\)-colourability seems ok.

The \((1, 1, 1)\)-colourability of such graphs follows from a result of Lih et al. ([2]) described in the following.

For every vertex \( v \in V \) let \( L(v) \) be a set (list) of available colours for the vertex \( v \).

**Definition 6.2.** A simple graphs \( G = (V, E) \) is called \((k, d)^*\)-list colourable if for every list assignment with \( |L(v)| = k \) for all \( v \in V \) we can choose a colour \( \varphi(v) \in L(v) \) such that for every colour \( i \) the maximum degree of \( G[V_i] \) is at most \( d \) where \( V_i = \{ v \in V | \varphi(v) = i \} \).

Lih et al. ([2]) proved that every planar graph without 4- and \( \ell \)-cycles for some \( \ell \in \{5, 6, 7\} \) is \((3, 1)^*\)-list colourable. Dong and Xu ([1]) proved an analogous result for planar graphs without 4- and \( \ell \)-cycles for some \( \ell \in \{8, 9\} \).

On the other hand there are planar graphs without 4- and 5-cycles which are not 3-list colourable \((3, 0)^*\)-list colourable) ([3]).

**References:**


7 \([1, 1, t]\)-COLOURINGS OF COMPLETE GRAPHS

(Arnfried Kemnitz, Massimiliano Marangio)

Given non-negative integers \(r\), \(s\), and \(t\), an \([r, s, t]\)-colouring of a graph \(G = (V(G), E(G))\) is a mapping \(c\) from \(V(G) \cup E(G)\) to the colour set \([0, 1, \ldots, k - 1]\) such that \(|c(v) - c(v')| \geq r\) for every two adjacent vertices \(v, v'\), \(|c(e) - c(e')| \geq s\) for every two adjacent edges \(e, e'\), and \(|c(v) - c(e)| \geq t\) for all pairs of incident vertices and edges, respectively. The \([r, s, t]\)-chromatic number \(\chi_{r,s,t}(G)\) of \(G\) is defined to be the minimum \(k\) such that \(G\) admits an \([r, s, t]\)-colouring.

This is an obvious generalization of all classical graph colourings since \(c\) is a vertex colouring if \(r = 1, s = t = 0\), an edge colouring if \(s = 1, r = t = 0\), and a total colouring if \(r = s = t = 1\), respectively. Therefore, \(\chi_{1,0,0}(G) = \chi(G), \chi_{0,1,0}(G) = \chi'(G),\) and \(\chi_{1,1,1}(G) = \chi''(G)\) where \(\chi(G)\) is the chromatic number, \(\chi'(G)\) the chromatic index, and \(\chi''(G)\) the total chromatic number of the graph \(G\).

For complete graphs \(K_n\) on \(n\) vertices it holds

\[
\chi_{1,1,1}(K_n) = \chi''(K_n) = \begin{cases} n & \text{if } n \text{ odd,} \\ n + 1 & \text{if } n \text{ even} \end{cases}
\]

and we proved (see [1] and [2])

\[
\chi_{1,1,2}(K_n) = \begin{cases} n & \text{if } n = 1, \\ n + 2 & \text{if } n \geq 3 \text{ odd, } n = 2, n = 6, \text{ or } n = 8, \\ n + 3 & \text{if } n = 4 \text{ or } n \geq 10 \text{ even,} \end{cases}
\]

\[
\chi_{1,1,t}(K_n) = 2n + t - 2 \text{ for } n \geq 2 \text{ and } t \geq n,
\]

\[
\chi_{1,1,t}(K_n) = n + 2t - 2 \text{ for } n \geq 3 \text{ and } t \geq 3 \text{ with } \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq t \leq n - 1.
\]

**Problem 7.1.** Is it true that \(\chi_{1,1,t}(K_n) = n + 2t + u\) if \(t < n\) where \(u\) is a small (positive or negative) constant (depending on \(n\) and \(t\))?

**Problem 7.2.** Determine \(\chi_{1,1,t}(K_n)\) for \(3 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor - 2\).

**References:**


8 RAINBOW CONNECTION AND SIZE OF GRAPHS

(Ingo Schiermeyer)

An edge-coloured connected graph $G$ is called *rainbow-connected* if each pair of distinct vertices of $G$ is connected by a path whose edges have distinct colours. The *rainbow connection number* of $G$, denoted by $rc(G)$, is the minimum number of colours such that $G$ is rainbow-connected. In [1] the following problem was introduced.

**Problem 8.1.** For all integers $n$ and $k$ with $1 \leq k \leq n - 1$ compute and minimize the function $f(n, k)$ with the following property: If $|V(G)| = n$ and $|E(G)| \geq f(n, k)$ then $rc(G) \leq k$.

In [1] the following lower bound for $f(n, k)$ has been shown.

**Proposition 8.1.** For $n$ and $k$ with $1 \leq k \leq n - 1$ it holds that $f(n, k) \geq \left(\frac{n-k+1}{2}\right) + k - 1$.

This lower bound is tight which can be seen by construction of a graph $G_k$ as follows: Take a $K_{n-k+1} - e$ and denote the two vertices of degree $n - k - 1$ with $u_1$ and $u_2$. Now take a path $P_k$ with vertices labeled $w_1, w_2, \ldots, w_k$ and identify the vertices $u_2$ and $w_1$. The resulting graph $G_k$ has order $n$ and size $|E(G_k)| = \left(\frac{n-k+1}{2}\right) + k - 2$. For its diameter we obtain $d(u_1, w_k) = \text{diam}(G_k) = k + 1 = rc(G_k)$.

**Problem 8.2.** Determine all values of $n$ and $k$ such that

$$f(n, k) = \left(\frac{n-k+1}{2}\right) + k - 1.$$ 

It has been shown that $f(n, k) = \left(\frac{n-k+1}{2}\right) + k - 1$ for

- $k = 1, 2, n - 2, \text{ and } n - 1$ in [1],
- for $k = 3$ and $4$ in [3],
- for $n - 6 \leq k \leq n - 3$ in [2].

**References:**


    Preprint 2012.

    Preprint 2011, Nankai University.
Lower bounds of special edge colourings

(Stephan Matos Camacho)

Consider a set $\mathcal{H}$ of binary vectors of length $l$. We define an addition $h_1 \oplus h_2$ for elements $h_1, h_2 \in \mathcal{H}$ in the following way:

$$(h_1 \oplus h_2)[i] := \begin{cases} 0 & \text{if } h_1[i] = h_2[i] = 0, \\ 1 & \text{if } h_1[i] = h_2[i] = 1, \\ 2 & \text{if } h_1[i] \neq h_2[i]. \end{cases}$$

Let $G = (V, E)$ be an edge-coloured graph, such that every $v \in V$ corresponds to a vector $h \in \mathcal{H}$, and two edges $e, f \in E$ with $e = v_1v_2$ and $f = v_3v_4$ are coloured with the same colour iff $v_1 \oplus v_2 = v_3 \oplus v_4$.

**Problem 9.1.** Determine $rd(G)$, the minimum number of colours occurring in such a graph $G$.

It can be easily computed, that $rd(K_2) = 1$, $rd(K_3) = 3$, $rd(K_4) = 5$, $rd(K_5) = 9$, $rd(K_6) = 12$, $rd(K_7) = 16$.

**Conjecture 9.1.**

$$rd(K_n) = \begin{cases} 3^k - 2^k & , \text{if } n = 2^k, \\ 3^k - 2^k - k & , \text{if } n = 2^k - 1, \\ 3^k & , \text{if } n = 2^k + 1. \end{cases}$$

This problem is motivated by a question concerning the Minimum Rainbow Subgraph problem ([1])

**References:**

10  **Polychromatic colourings of cube graphs**

(Heiko Harborth)

The polychromatic numbers $e(n,k)$ and $g(n,k)$ denote the maximum numbers of colours for the edges and for the vertices, respectively, of the $n$-dimensional cube graph such that each $k$-dimensional subcube graph contains all colours. See [1] for the limits of $e(n,k)$ and $g(n,k)$ if $k$ is fixed. - What about exact values of $e(n,k)$ and $g(n,k)$? See [2] for some first exact values of $g(n,k)$.

**References:**


11  **Saturated rainbow edge colouring of cube graphs**

(Heiko Harborth, Arnfried Kemnitz)

Let $f(n, k)$ denote the minimum number of colours for the edges of the cube graph $Q_n$ such that for $k < n$ no rainbow $Q_k$ occurs (the edges of a rainbow $Q_k$ have pairwise different colours), however, for every edge with a colour used at least twice it follows that a new colour for this edge induces a rainbow $Q_k$.

For $k = 3$ it is known $f(3, 3) = 11$, $f(4, 3) = 22$, and $f(5, 3) = 20$.

1. Determine $f(5, 3)$ and $f(6, 3)$.
2. Determine the smallest $n > 3$ such that $f(n, 3) = 11$.

**References:**

12 **EDGE-DISTINGUISHING INDEX OF A SUM OF CYCLES**

(Rafał Kalinowski, Mariusz Woźniak)

A *neighbourhood* $N(e)$ of an edge $e$ of a graph $G = (V, E)$ is a subgraph of $G$ induced by $e$ and all edges adjacent to $e$. A colouring $c : E \rightarrow S$ is called *edge-distinguishing* if, for any two distinct edges $e, e'$, there does not exist an isomorphism $\varphi$ of $N(e)$ onto $N(e')$ preserving colours of $c$, and such that $\varphi(e) = e'$. An *edge-distinguishing index* $\chi'_e(G)$ of a graph $G$ is the minimum number of colours in a proper edge-distinguishing colouring $c : E \rightarrow S$.

If $G$ is a cycle $C_n$ of length $n$, then it is easy to see that

$$\chi'_e(C_n) \geq \gamma_n := \min\{ k \mid \frac{1}{2}k^2(k - 1) \geq n \}.$$

**Theorem 12.1.**

$$\chi'_e(C_n) = \begin{cases} \gamma_n + 1 & \text{if } n = \frac{1}{2}k^2(k - 1) - 1 \text{ or } n = 4, \\ \gamma_n & \text{otherwise.} \end{cases}$$

**Problem 12.1.** Let $G$ be a disjoint sum of cycles with the total sum of lengths equal to $n$. Evaluate $\chi'_e(G)$. 

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16
13 Neighbours distinguishing index of planar graphs

(Keith Edwards, Mirko Horňák, Mariusz Woźniak)

Let $G$ be a finite simple graph with no component $K_2$. Let $C$ be a finite set of colours and let $\varphi : E(G) \rightarrow C$ be a proper edge colouring of $G$. The colour set of a vertex $v \in V(G)$ with respect to $\varphi$, in symbols $S_\varphi(v)$, is the set of colours of edges incident with $v$. The colouring $\varphi$ is neighbours distinguishing if $S_\varphi(x) \neq S_\varphi(y)$ for any $xy \in E(G)$. For example, any neighbour-distinguishing colouring of $C_5$ uses necessarily 5 colours.

The neighbours distinguishing index of the graph $G$ is the smallest number $\text{ndi}(G)$ of colours in a neighbour-distinguishing colouring of $G$. Neighbours distinguishing index has been introduced in [7], where the authors have conjectured that $\text{ndi}(G) \leq \Delta(G) + 2$ for any connected graph $G$ nonisomorphic to $C_5$ on at least three vertices (Neighbour-Distinguishing Conjecture = NDC). NDC was confirmed in [1] for cubic graphs and for bipartite graphs, in [6] for graphs with maximum degree at most 3, in [2] for planar graphs with girth at least 6 and in [5] for planar graphs with maximum degree at least 12. In [3] it was proved that $\text{ndi}(G) \leq \Delta(G) + 1$ for any planar bipartite graph $G$ with $\Delta(G) \geq 12$. Hatami in [4] showed that $\text{ndi}(G) \leq \Delta(G) + 300$ provided that $\Delta(G) > 10^{20}$.

**Problem 13.1.** Find the minimum integer $\Delta \geq 4$ such that $\text{ndi}(G) \leq \Delta(G) + 1$ for any plane bipartite graph $G$ with $\Delta(G) \geq \Delta$.

**Problem 13.2.** Prove or disprove NDC for planar graphs $G$ with $\Delta(G) = \Delta$ for (at least some) $\Delta \in \{4, \ldots, 11\}$.

References:


14 On the cyclic chromatic number of 3-connected plane graphs

(Mirko Horňák)

The cyclic chromatic number of a plane graph $G$, in symbol $\chi_c(G)$, is a minimum number of colours in such a vertex colouring of $G$ that distinct vertices incident with a common face receive distinct colours. If $G$ is 2-connected, then $\chi_c(G) \geq \Delta^*(G)$, where $\Delta^*(G)$ is the maximum face degree of $G$.

On the other hand, no 3-connected plane graph $G$ is known with $\chi_c(G) > \Delta^*(G) + 2$. Plummer and Toft ([7]) proved that $\chi_c(G) \leq \Delta^*(G) + 9$ and conjectured (PTC) that $\chi_c(G) \leq \Delta^*(G) + 2$ for any 3-connected plane graph $G$.

Let PTC($d$) denote PTC restricted to 3-connected plane graphs $G$ with $\Delta^*(G) = d$. It is known that PTC($d$) is true for $d = 3$ (Four Colour Theorem), $d = 4$ (Borodin [1]), $d \in \{18, \ldots, 23\}$ (Horňák and Zlámalová [6]) and $d \geq 24$ (Horňák and Jendrol’ [5]). For $\Delta^*(G) \geq 60$ Enomoto et al. ([4]) obtained the best possible inequality: $\chi_c(G) \leq \Delta^*(G) + 1$ (graphs of pyramids show that the bound $\Delta^*(G) + 1$ cannot be improved).

The best general upper bound known so far is due to Enomoto and Horňák ([3]), namely $\chi_c(G) \leq \Delta^*(G) + 5$.

Problem 14.1. Prove PTC($d$) for some $d \in \{5, \ldots, 17\}$.

References:


15 THE NEIGHBOUR-DISTINGUISHING-INDEX BY SUMS IN TOTAL PROPER COLOURINGS

(Monika Pilśniak, Mariusz Woźniak)

Let \(c : V \cup E \rightarrow \{1, 2, ..., k\}\) be a proper total coloring of a graph \(G\). For a vertex \(v\), we denote by \(f(v)\) the sum of colors of the edges incident to \(v\) and of the color of \(v\). The smallest \(k\) that guarantees that there is a coloring \(c\), so that the function \(f\) distinguishes adjacent vertices of \(G\), is called the total-neighbour-distinguishing-index by sums of \(G\), and it is denoted by \(\text{tndi}_{\Sigma}(G)\).

If we consider either trees and regular bipartite graphs, then \(\text{tndi}_{\Sigma}(G)\) equals to \(\Delta + 1\), if there does not exist two adjacent vertices of maximum degree, or \(\text{tndi}_{\Sigma}(G) = \Delta + 2\), otherwise.

For complete graphs, we can show that the total-neighbour-distinguishing-index depends on the parity of their order. Namely,

\[
\text{tndi}_{\Sigma}(K_n) = \begin{cases} 
  n + 1, & \text{if } n \text{ is even,} \\
  n + 2, & \text{if } n \text{ is odd.}
\end{cases}
\]

We can also show that, if \(G\) is a bipartite graph, or a cubic graph, or a graph with \(\Delta \leq 3\), then \(\text{tndi}_{\Sigma}(G) \leq \Delta + 3\). Hence, we may conjecture:

**Conjecture 15.1.** ([2]) For every graph \(G = (V, E)\), the total-neighbour-distinguishing-index by sums satisfies the inequality

\[
\text{tndi}_{\Sigma}(G) \leq \Delta + 3.
\]

In [3], the authors investigated also a proper total coloring of \(G\), but for every vertex \(v\) they assigned a set \(S(v)\) of colors of the edges incident to \(v\) and the color of \(v\). Similarly as above, by \(\text{tndi}(G)\) we can denote the smallest number \(k\) of colors, so that there exists a proper total coloring \(c\) for \(G\) and \(S(u)\) is different from \(S(v)\) for every pair of adjacent vertices \(u, v\).

**Conjecture 15.2.** ([3]) For every graph \(G = (V, E)\), the total-neighbour-distinguishing-index by sets \(\text{tndi}(G)\) satisfies the inequality

\[
\text{tndi}(G) \leq \Delta + 3.
\]

Zhang, Chen, Li, Yao, Lu and Wang considered the cases of cliques, paths, cycles, fans, wheels, stars, complete graphs, bipartite complete graphs and trees. They showed that \(\Delta + 3\) colors are enough in these cases. Next in [1] Chen proved this conjecture for bipartite graphs and for graphs with maximum degree at most three.

It is easy to observe, that if two vertices are distinguished by sums then they are also distinguished by sets, but not necessarily conversely.
References:


16  EVERY LOCALLY CONNECTED GRAPH IS WEAKLY PANCYCLIC

(Zdeněk Ryjáček)

Let $G$ be a finite simple undirected graph and let $g(G)$ and $c(G)$ be the girth and the circumference of $G$ (i.e. the length of a shortest cycle of $G$ and the length of a longest cycle of $G$), respectively. We say that $G$ is weakly pancyclic if $G$ contains cycles of all lengths $\ell$ for $g(G) \leq \ell \leq c(G)$. The graph $G$ is locally connected if the neighborhood of every vertex of $G$ induces a connected graph.

**Conjecture 16.1.** ([8]) Every connected locally connected graph is weakly pancyclic.

**Comments:** The concept of locally connected graphs was introduced by Chartrand and Pippert ([3]). More information about weakly pancyclic graphs appears in [2], for example.

The conjecture is based on a result by Clark ([4]), who proved that every connected, locally connected graph is vertex pancyclic (having cycles of all lengths from 3 to $|V(G)|$ through every vertex). Without the claw-free assumption, it is easy to construct locally connected graphs that are nonhamiltonian. Nevertheless, all known examples are weakly pancyclic; and indeed [4] proved the conjecture for claw-free graphs.

In a chordal graph, every block is locally connected, and for every cycle of length at least 4 there is a cycle with length one less that is obtained by skipping one vertex. Thus the conjecture holds for chordal graphs.

It is easy to show that the square of any graph is locally connected. (The square adds edges making vertices at distance 2 in the original graph adjacent.) Fleischner ([5], Theorem 6) proved that the square of every graph is weakly pancyclic, thus
verifying the conjecture for squares of graphs.

The lexicographical product of graphs is another way to obtain a locally connected graph. Kaiser and Kriesell ([6]) recently proved that the lexicographical product $G[H]$ is weakly pancyclic provided $G$ is a connected graph and $H$ is an arbitrary graph with at least one edge.

Kriesell ([7]) verified the conjecture for graphs with maximum degree at most 4.

Finally, planar triangulations are locally connected. Balister ([1]) proved the conjecture for this class as follows. Let $C$ be a cycle in a planar triangulation $G$. By induction on the number of faces inside, we prove that the interior (with boundary) contains cycles of all shorter lengths. If some face inside has two edges on $C$, then using the third edge yields a cycle $C'$ with length one less and fewer faces inside. Otherwise, there is a face with one edge on $C$ and the third vertex inside. Detouring from $C$ to include this vertex forms a longer cycle $C'$, but again it has fewer regions inside and the induction hypothesis applies.

References:


J. Graph Theory 27 (1998), 141-176.


[4] L. Clark, Hamiltonian properties of connected locally connected graphs

[5] H. Fleischner, In the square of graphs, hamiltonicity and pancyclicity, hamiltonian connectedness and panconnectedness are equivalent concepts
Monatshefte für Mathematik 82 (1976), 125-149.

Graphs Comb. 22 (2006), 51-58.


[8] Z. Ryjáček, Weak pancyclicity of locally connected graphs
17 CONNECTED EVEN FACTORS IN LOCALLY CONNECTED GRAPHS

(Zdeněk Ryjáček and Liming Xiong)

All graphs are finite, simple and undirected. We use \( d_G(x) \) for the degree of a vertex \( x \in V(G) \), \( \Delta(G) \) for the maximum degree of \( G \), \( N_G(x) \) for the set of neighbors of a vertex \( x \) in \( G \), and \( \alpha(G) \) for the independence number of \( G \). For \( M \subset V(G) \), we use \( G[M] \) to denote the induced subgraph of \( G \) on \( M \), and we say that \( G \) is locally connected if \( G[N_G(x)] \) is a connected graph for every \( x \in V(G) \). A graph \( G \) is hamiltonian if \( G \) has a spanning cycle, vertex pancyclic if \( G \) has cycles of all lengths from 3 to \( |V(G)| \) through every vertex, and weakly pancyclic if \( G \) has cycles of all lengths from 3 to the circumference of \( G \). A spanning connected subgraph of \( G \) in which every vertex has even degree is called a connected even factor (or an eulerian factor) of \( G \). For \( s \geq 1 \), if \( F \) is a connected even factor of \( G \) such that \( \Delta(F) \leq 2s \), we say that \( F \) is a connected even \([2, 2s]\)-factor of \( G \). Finally, for \( r \geq 3 \), \( G \) is \( K_{1,r}\)-free if \( G \) contains no induced subgraph isomorphic to \( K_{1,r} \), and for \( r = 3 \) we say that \( G \) is claw-free.

Our problem is motivated by the following two well-known results.

**Theorem 17.1.** Oberly and Sumner, [2]) Every connected, locally connected claw-free graph of order at least 3 is hamiltonian.

**Theorem 17.2.** (Clark, [1]) Every connected, locally connected claw-free graph of order at least 3 is vertex pancyclic.

Now, one can ask what can be obtained without the condition that \( G \) is claw-free. The following was conjectured in [3] (see also Problem 16 of this brochure).

**Conjecture 17.1.** (Ryjáček, [3]) Every connected, locally connected graph of order at least 3 is weakly pancyclic.

It is easy to observe that a graph \( G \) is claw-free if and only if \( \alpha(G[N_G(x)]) \leq 2 \) for every vertex \( x \in V(G) \), and, more generally, \( G \) is \( K_{1,r}\)-free \( (r \geq 3) \) if and only if \( \alpha(G[N_G(x)]) \leq r - 1 \) for every \( x \in V(G) \). Thus, the following conjecture, if true, gives another generalization of Theorem 17.1.

**Conjecture 17.2.** Every connected, locally connected graph \( G \) of order at least 3 has a connected even factor \( F \) such that \( d_F(x) \leq \alpha(G[N_G(x)]) + 1 \) for every vertex \( x \in V(G) \).

So far, the best known result in the direction of Conjecture 17.1 is the following.
**Theorem 17.3.** (Yin and Xiong, [4]) Every connected, locally connected $K_{1,s+2}$-free graph of order at least 3 has a connected even $[2, 2s]$-factor.

The following weaker version of Conjecture 17.2, if true, gives also a generalization of Theorem A.

**Conjecture 17.3.** (Yin and Xiong, [4]) Every connected, locally connected graph $K_{1,2s+1}$-free graph of order at least 3 has a connected even $[2, 2s]$-factor.

**References:**


[2] D. J. Oberly and D. P. Sumner, Every connected, locally connected nontrivial graph with no induced claw is hamiltonian, J. Graph Theory 3 (1979), 351-356.


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**18 Dominating cycles and Hamiltonian prisms**

(Zdeněk Ryjáček)

The *prism over a graph* $G$, denoted $G\Box K_2$, is the Cartesian product of $G$ and $K_2$. It consists of two disjoint copies of $G$ and a perfect matching connecting a vertex in one copy of $G$ to its “clone” in the other copy.

A graph $G$ is *hamiltonian* if it has a hamiltonian cycle and *traceable* if it has a hamiltonian path. Define a $k$-walk in a graph to be a spanning closed walk in which every vertex is visited at most $k$ times.

The following implications are easy to verify:

$$G \text{ is hamiltonian } \Rightarrow G \text{ is traceable } \Rightarrow G\Box K_2 \text{ is hamiltonian } \Rightarrow G \text{ has a 2-walk.}$$

Thus the question whether $G\Box K_2$ is hamiltonian is “sandwiched” between hamiltonicity and having a 2-walk. Specifically, the property of having a hamiltonian prism can be considered as a “relaxation” of hamiltonicity. More information about prism-hamiltonicity of a graph can be found e.g. in [1] and [2].
A dominating cycle in a graph \( G \) is a cycle \( C \) such that every edge of \( G \) has at least one vertex on \( C \), i.e., such that the graph \( G - C \) is edgeless. Clearly, a hamiltonian cycle is dominating, and hence the property of having a dominating cycle can be considered as another relaxation of hamiltonicity.

There is a natural question whether there is any relation between these two properties.

**Example 18.1.** Let \( H \) be any 2-connected cubic nonhamiltonian graph, and let \( G \) be obtained from \( H \) by replacing every vertex of \( H \) with a triangle (such a \( G \) is sometimes called the inflation of \( H \)). Then \( G \) is a 2-connected line graph and these are known [2] to be prism-hamiltonian. On the other hand, since \( H \) is nonhamiltonian, any cycle in \( G \) has to miss at least one "new" triangle and hence \( G \) has no dominating cycle. Thus, there are "many" graphs showing that hamiltonian prism does not imply having a dominating cycle.

**Example 18.2.** The graph in the figure below shows that also the existence of a dominating cycle does not imply having hamiltonian prism.

![Graph Example](image)

However, all such known examples are of low toughness. This motivates the following question.

**Conjecture 18.1.** Let \( G \) be a 1-tough graph having a dominating cycle. Then \( G \) has hamiltonian prism.

**Comments:** Recall that \( G \) is 1-tough if, for any \( S \subset V(G) \), the graph \( G - S \) has at most \( |S| \) components.

Suppose that \( G \) has a dominating cycle \( C \) of even length. Set \( M = V(G) \setminus V(C) \) and \( N = \{ x \in V(C) \mid x \text{ has a neighbor in } M \} \). Then the graph induced by \( M \cup N \) has a matching containing all vertices from \( M \) (this follows by the toughness assumption and by the Hall’s theorem). Using this matching, it is easy to construct a hamiltonian cycle in \( G \square K_2 \).

The difficult case is when all dominating cycles in \( G \) are of odd length.
References:


J. Graph Theory 56 (2007), 249-269.

19 Nonpancylic claw-free graphs with complete closure

(Zdeněk Ryjáček, Richard Schelp)

It is known that a claw-free graph $G$ is hamiltonian if and only if its closure $\text{cl}(G)$
is hamiltonian. On the other hand, there are nonpancyclic graphs with pancyclic closure [1]. The graph in the figure below is an example of such a nonpancyclic graph with complete (and hence pancyclic) closure.

![Graph example](image)

Problem 19.1. Determine the maximum number of cycle lengths that can be missing in a claw-free graph on $n$ vertices with complete closure.

It is easy to see that a claw-free graph with complete closure on at least 4 vertices can miss neither a $C_3$ nor a $C_4$. The main result of [2] shows that such a graph $G$ cannot be missing a cycle of length $n - 1$; however, the proof of this result is difficult and cannot be iterated.

The following was conjectured in [2].

Conjecture 19.1. Let $c_1, c_2$ be fixed constants. Then for large $n$, any claw-free graph $G$ of order $n$ whose closure is complete contains cycles $C_i$ for all $i$, where $3 \leq i \leq c_1$ and $n - c_2 \leq i \leq n$. 
Recently, counterexamples to the first part of the Conjecture 19.1 have been found (see [3]), all these counterexamples have connectivity $\kappa \leq 5$. We believe that the second part of Conjecture 19.1 is true, and that such a construction as shown in [3] is possible only for connectivity $\kappa \leq 5$. Thus, we conjecture the following.

**Conjecture 19.2.** Let $c$ be a fixed constant. then for large $n$, any claw-free graph $G$ of order $n$, whose closure is complete, contains cycles $C_i$ for all $i$, $n - c \leq i \leq n$.

**Conjecture 19.3.** Every 6-connected claw-free graph with complete closure is pancyclic.

**References:**


## 20 Hamiltonian Neighborhood Graphs

(Martin Sonntag, Hanns-Martin Teichert)

The *neighborhood graph* $N(G)$ for a simple graph $G = (V, E)$ is defined to be the graph on the same vertex set $V$ with two vertices adjacent if and only if there is in $G$ a path of length two between them. Neighborhood graphs, also referred to as *two-step graphs*, have been the object of several studies in the last 25 years.

If $G$ is hamiltonian then $N(G)$ is hamiltonian for $|V|$ odd. This is not true for $|V|$ even, for instance $N(C_{2n})$ is disconnected. We can show that $N(G)$ is always hamiltonian if $G$ is 1-hamiltonian connected and has a triangle, but we think there are weaker conditions providing hamiltonicity of $N(G)$.

**Problem 20.1.** Find sufficient conditions for $G$, such that $N(G)$ is hamiltonian or hamiltonian connected.
21 Spanning connectivity

(Elkin Vumar)

Let $G = (V, E)$ be a connected graph. For $u, v \in V(G)$, a $k^*$-container $C(u, v)$ is the set of $k$ internally disjoint $(u, v)$-paths that contains all vertices of $G$. $G$ is $k^*$-connected if there is a $k^*$-container between any distinct pair of vertices. By definition, $G$ is $1^*$-connected if and only if it is Hamilton-connected, and $G$ is $2^*$-connected if and only if it is Hamiltonian.

Problem 21.1. Determine the minimum number $f = f(k, \kappa, d)$ such that the $f$-th power $G^f$ of a graph $G$ with connectivity $\kappa \geq 1$ and diameter $d$ is $k^*$-connected. In particular, prove or disprove that $G^2$ is $3^*$-connected for a 2-connected graph $G$.

Some known results.

1. $C_n^2$ is $3^*$-connected for an $n$ cycle $C_n (n \geq 3)$ [1]. Hence $G^2$ is $3^*$-connected for a hamiltonian graph $G$.

2. If $G$ is a connected graph with $|V(G)| \geq k + 1 \geq 4$, then $G^k$ is $k^*$-connected [2].

References:


22 Arbitrarily vertex-decomposable trees

(Mirko Horňák, Antoni Marczyk and Mariusz Woźniak)

A tree $T$ is said to be arbitrarily vertex decomposable if for any sequence $(t_1, \ldots, t_k)$ of positive integers adding up to $|V(T)|$ there is a sequence $(T_1, \ldots, T_k)$ of vertex-disjoint subtrees of $T$ such that $|V(T_i)| = t_i$ for $i = 1, \ldots, k$.

The notion of an arbitrarily vertex decomposable (avd for short) tree has been introduced independently by Barth et al. in [1] and Horňák and Woźniak in [5].

It turned out that some classes are essential when analysing the property of a tree “to be avd”. A star-like tree (a spider) is a tree homeomorphic to a star $K_{1,q}$. Such a tree is uniquely (up to isomorphism) determined by the non-decreasing sequence $(a_1, \ldots, a_q)$ of orders of its arms; it will be denoted by $S(a_1, \ldots, a_q)$ and also called a $q$-spider.

A caterpillar is a tree $T$ having as a subgraph a path $P$ such that $T - P$ is an edgeless graph.

The most general result concerns the best upper bound $\text{avd}_{\text{max}}$ on the maximum degree of an avd tree. In [5] it has been conjectured that $\text{avd}_{\text{max}} \leq 6$ and conjectured that $\text{avd}_{\text{max}} = 4$. Later it was shown that $\text{avd}_{\text{max}} \leq 5$ ([7]) and $\text{avd}_{\text{max}} \leq 4$ ([2]). More precisely, the result of [2] reads as follows:

**Theorem 22.1.** If a tree $T$ is avd, then $\Delta(T) \leq 4$. Moreover, if a tree $T$ is avd, then each vertex of $T$ of degree four is adjacent to a leaf.

Let us mention that there are avd trees with maximum degree 4, for example $S(2, 2, 5, 7)$, hence, as conjectured, $\text{avd}_{\text{max}} = 4$.

There are only few known families of avd trees. The following theorem has been proved independently in [1] and [5] (see also [4] for another, much more complicated result of this type).

**Theorem 22.2.** A 3-spider $S(2, a_2, a_3)$ is avd if and only if $a_2$ and $a_3$ are coprime.

For $a_1 \geq 2$ let $A_2(a_1)$ be the set of all $a_2$’s such that $a_2 \geq a_1$ and there is $a_3 \geq a_2$ such that $S(a_1, a_2, a_3)$ is avd. Similarly, for $a_2 \geq a_1 \geq 2$ let $A_3(a_1, a_2)$ be the set of all $a_3$’s such that $a_3 \geq a_2$ and there is $a_4 \geq a_3$ such that $S(a_1, a_2, a_3, a_4)$ is avd. From Theorem 22.1 it is clear that if $A_3(a_1, a_2) \neq \emptyset$, then $a_1 = 2$.

We give below four “main” open questions concerning avd trees.

**Question 22.1.** Is $A_2(a_1) \neq \emptyset$ for all $a_1 \geq 2$?

**Question 22.2.** Is $A_3(2, a_2) \neq \emptyset$ for all $a_2 \geq 2$?
Horňák and Woźniak ([6]) showed that $A_2(\alpha_1) \neq \emptyset$ for all $\alpha_1 \in \{2, \ldots, 28\}$ and $A_3(2, \alpha_2) \neq \emptyset$ for all $\alpha_2 \in \{2, \ldots, 23\}$. According to [3] there are infinitely many $\alpha_1$’s such that $A_2(\alpha_1) \neq \emptyset$.

It is easy to see that an avd caterpillar has at most one vertex of degree four. In Figure 1 there is depicted an avd tree having two vertices of degree four.

References:


23 Twins in graphs and sequences

(Maria Axenovich)

For a combinatorial structure $G$ and a set of parameters, we say that two disjoint substructures are twins if those parameters coincide for each substructure. This notion generalizes the pigeonhole principle and was investigated in case of sequences and graphs.

Sequences

In the case of sequences it is known, see [2], that any binary sequence of length $n$ contains two disjoint identical subsequences of length $n/2 - o(n)$ each.

**Open Problem 1.** Let $f(n, 3)$ be the largest $k$ such that any sequence over an alphabet with three letters contains two identical disjoint subsequences of length $k$ each. It is known that $f(n, 3) \geq n/3 - o(n)$. Is it true that $f(n, 3) = n/2 - o(n)$?

Graphs

The following twin problem in graphs was introduced by Caro and Yuster in [3]. For a graph $G$, we call two disjoint subsets of vertices twins if they have the same size and induce subgraphs with the same number of edges. Let $t(G)$ be the largest $k$ such that there are twins $A, B$ in $G$ with $|A| = |B| = k$. Let

$$t(n) = \min\{t(G) : |V(G)| = n\}.$$

The best currently known bounds on $t(n)$ are given in the following theorem.

**Theorem 23.1.** (Caro and Yuster [3])

There exists a positive constant $c$ such that $\sqrt{n} \leq t(n) \leq n/2 - c \log \log n$.

In addition, there is a version of this result for sparse graphs:

**Theorem 23.2.** (Caro and Yuster [3])

For every fixed $\alpha > 0$ and for every $\epsilon > 0$ there exists $N = N(\alpha, \epsilon)$ so that for all $n > N$, if $G$ is a graph on $n$ vertices and at most $n^2 - \alpha$ edges then $t(G) \geq (1 - \epsilon)n/2$.

In [1], we proved the following:

**Theorem 23.3.** If $G$ is a graph on $n$ vertices and $e$ edges then $t(G) \geq \frac{n}{2} \left(1 - \frac{20 \sqrt{e \log n}}{n}\right)$.

This theorem implies, in particular, that any $n$-vertex graph on $o\left(n^2 / \log^2 n\right)$ edges has twins of size $n/2 - o(n)$ each and that planar graphs have twins of size at least $n/2 - c \log n$ each.
Theorem 23.4. Let $G$ be a graph on $n$ vertices, where $n$ is even. Let $V_i$ be the set of vertices of degree $i$, $i = 0, \ldots, n - 1$. If one of the following conditions 1-4 holds then $t(G) = n/2$.

1. The degree sequence of $G$ forms a set of consecutive integers.
2. $|V_i|$ is even for each $i$.
3. $n \geq 90$ and $|\{i : |V_i| \text{ is odd}\}| > n/2$.
4. There are at least $\Delta(G) - \delta(G)$ disjoint consecutive pairs of vertices.

Theorem 23.5. If $G$ is a forest then $t(G) \geq \lceil n/2 \rceil - 1$.

If $n$ is odd this is clearly best possible. For even $n$ this is attained, for example, by a star. Such a graph has no perfect twins.

Open Problem 2. Improve the bounds on $t(n)$.

Open Problem 3. Give an easier proof for the twin result in forests.

References:


24 Decomposing Bipartite Graphs into Locally Irregular Subgraphs

(Olivier Baudon, Julien Bensmail, Jakub Przybyło, Mariusz Woźniak)

A locally irregular graph is a graph whose adjacent vertices have distinct degrees. We say that a graph \( G \) can be decomposed into \( k \) locally irregular subgraphs if its edge set may be partitioned into \( k \) subsets each of which induces a locally irregular subgraph in \( G \). This is equivalent to painting the edges of \( G \) with \( k \) colours so that every colour class induces a locally irregular subgraph in \( G \). Such property of a \( k \)-edge colouring is stronger than the one investigated by Addario-Berry et al. in [1], who required the neighbours in \( G \) to be incident with distinct multisets of colours. This problem is also related to the ‘1-2-3 Conjecture’, see e.g. [3]. It is known that all connected graphs except for odd paths and cycles and a special family of graphs of maximum degree 3 can be decomposed into (a certain number of) locally irregular subgraphs, and it has been conjectured that each of these can be decomposed into (at most) three such subgraphs, see [2]. This has been verified e.g. for trees, complete graphs, complete bipartite graphs and for graphs of sufficiently large minimum degree, see [2] and [4].

Apart from solving the main conjecture entirely, the following problem and its weaker counterpart below seem to be intriguing.

**Conjecture 24.1.** Every connected bipartite graph which is not an odd length path can be decomposed into three locally irregular subgraphs.

**Conjecture 24.2.** There exists a constant \( K \) such that every connected bipartite graph which is not an odd length path can be decomposed into (at most) \( K \) locally irregular subgraphs.

Finally, one might also consider a similar weaker correspondent of the main conjecture itself.

**References:**


While investigating the minimum order of $k$–chromatic $K_{r+1}$–free graphs the following question concerning Ramsey numbers arises:

**Question 25.1.** Let $G$ be a graph with clique number at most $r$, independence number 3, and order $n = R(r+1, 3) + 1$. Do there always exist two vertex-disjoint independent sets with 3 vertices?

**Known:**

- If the order of $G$ is $n = R(r+1, 3) + 2$, then the answer is “yes” ([1]).

- If $G$ is a graph on at least 10 vertices then either $G$ contains a clique or an independent set on four vertices, or $G$ contains two disjoint triangles ([2]). This affirms the question for $r = 3$.

**References:**

Preprint 2010-01 (2010), TU Bergakademie Freiberg.

Habilitation thesis (2009), Université Paris.
Dominating Sets

26 DOMINATION HYPERGRAPHS OF TOURNAMENTS

(Martin Sonntag, Hanns-Martin Teichert)

Let $D = (V, A)$ be a digraph. A subset $V' \subseteq V$ is called a dominating set iff $\forall x \in V \setminus V' \exists y \in V': (y, x) \in A$. The domination graph $D(D)$ has vertex set $V$ and its edges are the dominating sets of cardinality two (see for instance [1]). As a natural generalization the domination hypergraph $DH(D)$ also has vertex set $V$ and its edges are all minimal dominating sets $V'$ with $|V'| \geq 1$.

There are many interesting results on domination graphs of tournaments $T_n$, e.g. in general $D(T_n)$ is not connected (see for instance [2]).

Conjecture 26.1. The domination hypergraph $DH(T_n)$ of a tournament $T_n$ consists of at most one nontrivial connected component.

The conjecture is true for $n \leq 9$. We tested hundreds of bigger examples (up to $n = 23$) by Mathematica routines and found no counterexample. It is easy to prove that every nontrivial component of $DH(T_n)$ contains at least three edges. A first step to verify the conjecture could be the investigation of regular tournaments.

References:


Properties of Graphs

27 Weight of graphs having a given property

(Stanislav Jendrol')

Let $G$ be a graph of positive size and let $e = xy$ be an edge of $G$. The weight of $e$ is $w(e) := \deg_G(x) + \deg_G(y)$ and the weight of $G$ is $w(G) := \min(w(e) : e \in E(G))$. Let $n, m \in \mathbb{Z}$, $n \geq 2$, $1 \leq m \leq \binom{n}{2}$, let $P$ be a graph property and let $P(n, m)$ be the set of all graphs in $P$ of order $n$ and size $m$. If $P(n, m) \neq \emptyset$, we define $w(n, m, P) := \max(w(G) : G \in P(n, m))$. In the case $P = \mathcal{I}$ (the most general property “to be a graph”) the problem of determining $w(n, m, \mathcal{I})$ was formulated by Erdős during the conference in Prachatice held in 1990 and solved by Jendrol' and Schiermeyer in [1].

We have obtained the results mentioned below.

Let $\mathcal{B}$ denote bipartite graphs. For a pair $(n, m)$ such that $\mathcal{B}(n, m) \neq \emptyset$ let $a, b, s, p, w$ be integers defined by

- $a := \lceil n - \sqrt{n^2 - 4m} \rceil$,
- $b := \lceil \frac{m}{a} \rceil$,
- $s := ab - m$,
- $p := \min(s, 2)$

and $w := a + b - p$.

**Theorem 27.1.** If $n \geq 2$ and $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$, then the following hold:

1. $1 \leq w \leq w(n, m, \mathcal{B}) \leq w + 1 \leq n$.
2. If $w(n, m, \mathcal{B}) = w + 1$, then $w(n, m, \mathcal{B}) = n$.
3. $w(n, m, \mathcal{B}) = n$ if and only if $\sqrt{n^2 - 4m}$ is an integer.
4. $w(n, m, \mathcal{B}) = n - 1$ if and only if one of the numbers $\sqrt{n^2 - 4m - 4}$ and $\sqrt{(n - 1)^2 - 4m}$ is an integer.
5. If $\sqrt{n^2 - 4m}$, $\sqrt{n^2 - 4m - 4}$ and $\sqrt{(n - 1)^2 - 4m}$ are not integers, then $w(n, m, \mathcal{B}) = w$.

Let $\mathcal{C}$ denote connected graphs and $\mathcal{D}_{1+}$ graphs with minimum degree at least 1. For a pair $(n, m)$ such that $\mathcal{C}(n, m) \neq \emptyset$ let $k, m', r, c, d, e$ be integers defined by
\(\binom{n}{2} - \binom{k+1}{2} < m \leq \binom{n}{2} - \binom{k}{2}\), \(m' := \binom{n}{2} - \binom{k}{2} - m\), \(r := \left\lfloor \frac{m'}{n-k} \right\rfloor\), \(c := 1\) if either \(m' \leq \left\lfloor \frac{n-k}{2} \right\rfloor\) or \(m' = (n-k-1)^2\), \(c := 2\) otherwise, \(d := \left\lfloor \frac{2m}{n} \right\rfloor\), \(e := 0\) if either \(m = \binom{n}{2} - 1\) or \(d \leq n-3\) and \(2m \equiv q \pmod{n}, 0 \leq q \leq d - 1\), and \(e := 1\) otherwise.

Theorem 27.2. If \(48 \leq n - 1 \leq m \leq \binom{n}{2} - \binom{\lceil n/2 \rceil}{2}\), \(k \geq n/2\) and \(C \subseteq P \subseteq D_1\), then \(w(n, m, P) = 2n - k - r - c\).

Theorem 27.3. If \(n \geq 15, \binom{n}{2} - \binom{\lceil n/2 \rceil}{2} + 1 \leq m \leq \binom{n}{2}\) and \(C \subseteq P \subseteq D_1\), then \(w(n, m, P) = 2d + e\).

Problem 27.1. Complete the results of Theorem 27.2 for \(n \leq 48\) and those of Theorem 27.3 for \(n \leq 14\).

Problem 27.2. Find \(w(n, m, P)\) for other graph properties \(P\).

References:

28 Turan type numbers for edges in hypercube graphs

(Heiko Harborth, Hauke Nienborg)

Let \(f(n, k, s)\) denote the maximum number of edges of \(Q_n\) such that at least \(s\) edges of every subgraph \(Q_k\) are missing. Thus \(f(n, k, 1)\) is the Turan type problem of \(P\). Erdős for subgraphs \(Q_k\) of the host graph \(Q_n\). – Especially, determine \(f(n, k, k^2(k-1) - 1) = E(n, k)\) being the maximum number of edges of \(Q_n\) such that every subgraph \(Q_k\) contains at most one edge.

References:
29 **MINIMUM NUMBER OF CROSSINGS OF PSEUDO DIAGONALS IN CONVEX POLYGONS**

(Heiko Harborth)

Consider convex \( n \)-gons with all diagonals as pseudoline segments. Two diagonals with a common vertex do not intersect and two disjoint diagonals intersect at most once. Determine the minimum number \( c(n) \) of crossings if multiple crossings are allowed. So, \( c(7) = 29, c(8) = 49, c(9) = 86, c(10) = 126, c(11) \leq 198, c(12) \leq 273, c(14) \leq 526, \) and \( c(16) \leq 834 \) instead of \( 1820 = \binom{16}{4} \).

30 **MULTIPLE CROSSINGS IN DRAWINGS OF COMPLETE GRAPHS**

(Heiko Harborth)

Let \( M(2m) \) denote the maximum number of \( m \)-fold crossings of edges in drawings of the complete graph \( K_{2m} \) (at most one crossing for disjoint edges and no crossings for edges with a common vertex). Prove that \( M(2m) \leq 2 \) for \( m \geq 5 \).

References:


31 **CROSSING REGULAR CYCLE DRAWINGS**

(Heiko Harborth)

Crossing \( r \)-regular drawings \( D(G) \) are realizations of a graph \( G \) in the plane where the curves for its edges have at most one point in common, either a crossing or an endpoint and where every edge has exactly \( r \) crossings. Do crossing \( r \)-regular drawings \( D(C_n) \) of the cycle graph \( C_n \) exist for \( n \leq 5 \) if \( n \) is even for \( 0 \leq r \leq n - 3 \), and if \( n \) is odd for \( 0 \leq r \leq n - 3 \), \( r \) odd? For \( r \leq 9 \) constructions are known in all cases.
Properties of Graphs

References:

32 Ramsey numbers for graph drawings

(Heiko Harborth)

The drawing Ramsey number Dr(G) is the smallest r such that every drawing of
the complete graph Kr in the plane (two edges have at most one point in common,
either a crossing or a vertex) contains at least one subdrawing of G having its
maximum number of crossings CR(G).
For G = Ks the numbers Dr(K3) = 3 and Dr(K4) = 5 are trivial. Determine
Dr(K5). It is known 10 \leq Dr(K5) \leq 113.

References:

33 Sphere graphs or Kuratowski in R^3

(Heiko Harborth)

In 1930 K. Kuratowski proved the graph planarity criterion: A graph is planar if and
only if it does not contain a subgraph homeomorphic to either K5 or K3,3.
Planar graphs also are known to be equivalent to coin graphs, that is, to nonover-
lapping circles around the vertexpoints such that two circles have a touching point
if and only if the corresponding vertices are adjacent.
What about spatial graphs, that is, there are nonoverlapping spheres around the
vertexpoints such that two spheres have a touching point if and only if the corre-
sponding vertices are adjacent.
Find a criterion for spatial graphs corresponding to Kuratowski’s criterion for planar
graphs. Are K6 and K4,4 the graphs corresponding to the Kuratowski graphs K5 and
K3,3?
34  **Straight ahead cycles in drawings of eulerian graphs**

(Heiko Harborth)

A path in a drawing $D(G)$ of a graph $G$ (two edges have at most one point in common, either a crossing or a vertex) passes a vertex straight ahead if it leaves the same number of edges on both sides.

**Problem 34.1.** For any Eulerian graph $G$ does there exist a $D(G)$ inducing an Eulerian straight ahead cycle? - Examples are known for up to 8 vertices, for complete graphs $K_{2^r+1}$, and for complete bipartite graphs $K_{2^r,2^s}$. What about cube graphs $Q_{2d}$ for $d \geq 3$?

**Problem 34.2.** For $G = K_{2^r+1}$ with $2^r + 1 \equiv 1 \text{ or } 3 \pmod{6}$ does there exist a $D(K_{2^r+1})$ inducing straight ahead triangles only? - An example is known for $K_7$. Find a $D(K_9)$. 
Paths

35 Minimum K-path vertex cover

(Jan Katrenič, Ingo Schiermeyer, Gabriel Semanišin)

Let $G$ be a graph and $k \geq 2$ be an arbitrary, fixed integer. Then $S \subseteq V(G)$ is called a $k$-path vertex cover of $G$ if every path on $k$ vertices in $G$ contains a vertex from $S$. The cardinality of a minimum $k$-path vertex cover of $G$ is denoted by $\psi_k(G)$. Note that $\psi_2(G)$ is the cardinality of a minimum vertex cover of $G$.

The concept of $k$-path vertex cover is related to the algorithm of M. Novotný for a secure communication in wireless sensor networks (see [4]). The invariant $\psi_k(G)$ was introduced in [1] and some exact values and estimations of it are presented in [1], [2]. For example, for $d$-regular graphs we have the following result:

**Theorem 35.1.** Let $k \geq 2$ and $d \geq 1$ be positive integers. Then for any $d$-regular graph $G$ the following holds:

$$\psi_k(G) \geq \frac{d - k + 2}{2d - k + 2}|V(G)|.$$  

An associated $k$-Path Vertex Cover Problem (abbreviated by $k$-PVCP) can be formulated in the following way:

**Problem 35.1.** Given a graph $G$ and a positive integer $k$. Find a minimum $k$-path vertex cover of $G$.

Some algorithmic aspects of the $k$-PVCP were studied in [1]. It was proved there that the $k$-PVCP is NP-hard for any fixed integer $k \geq 2$. On the other hand, the value of $\psi_k(G)$ for trees and $\psi_3(G)$ for outerplanar graphs can be computed in linear time. One can easily see, that for an arbitrary positive integer $k$, there is a trivial $k$-approximation of $k$-PVCP. The following non-deterministic improvement is obtained in [3]:

**Theorem 35.2.** For the 3-PVCP there is a deterministic algorithm DPVC of approximation ratio $\max(2, \frac{5}{2} \cdot \frac{\Delta - 1}{\Delta + 1})$, which runs in time $O(2^\Delta n^{O(1)})$ on a graph $G$ with $n$ vertices and maximum degree $\Delta$.  

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A deterministic approach was presented in [6] and [7]:

**Theorem 35.3.** There is a deterministic 2-approximation algorithm for 3-PVCP.

Some algorithms for a modification of the original problem were studied in [5]. The presented results lead us to the following problem:

**Problem 35.2.** Let $k \geq 4$. Is there a $k - 1$ factor approximation algorithm for the $k$-PVCP?

References:


36 THE VERTEX COVER $P_k$ PROBLEM

(Jianhua Tu)

Let $G = (V, E)$ be a graph and let $k$ be a positive integer. A subset of vertices $F \subseteq V$ is called a vertex cover $P_k$ set if every path of order $k$ in $G$ contains at least one vertex from $F$. The vertex cover $P_k$ number of $G$, denoted by $\psi_k(G)$, is the cardinality of a minimum vertex cover $P_k$ set. The vertex cover $P_k$ ($VCP_k$) problem is to find a minimum vertex cover $P_k$ set. Clearly, the $VCP_2$ problem corresponds to the well-known vertex cover problem.

In [7,8], Tu et al. proved that the $VCP_k$ problem is NP-hard for any integer $k \geq 2$ and gave two 2-approximation algorithms for the $VCP_3$ problem (for the weighted case of the problem) using the primal-dual method and the local-ratio method. Brešar et al. [2] presented a linear-time algorithm for the $VCP_k$ problem for trees. Kardos et al. [5] presented an exact algorithm with a running time of $O(1.571^n)$ for the $VCP_3$ problem on a graph with $n$ vertices. In addition, some bounds on $\psi_k(G)$ have been given in [1,2].

**Theorem 36.1** ([4]). **(Lovász decomposition)** For any graph $G$ of maximum degree $\Delta$, the vertex set of $G$ can be partitioned into $k$ sets which induce subgraphs of maximum degree at most $\frac{\Delta}{k}$. Moreover, this can be computed in running time $O(\Delta|E(G)|)$.

We have proven the $VCP_3$ problem and the $VCP_4$ problem are NP-hard for the cubic planar graphs. Further, we have the following results.

**Theorem 36.2** ([9]). We can use the Lovász decomposition to get a 1.25-approximation algorithm for the $VCP_3$ problem in cubic graphs.

**Theorem 36.3** ([10]). We can use the Lovász decomposition to get a 2-approximation algorithm for the $VCP_4$ problem in cubic graphs.

**Problem 36.1.** For the weighted $VCP_3$ problem and the weighted $VCP_4$ problem, are there better approximation algorithms in the cubic graphs.

Nemhauser and Trotter [6] proved a famous theorem for the vertex cover problem in combinatorial optimization, it can be formulated as follows:

**Theorem 36.4** ([6]). For an undirected graph $G = (V, E)$, one can compute in polynomial time two disjoint vertex subsets $A$ and $B$ such that the following three properties hold:

1. If $S'$ is a vertex cover of the induced subgraph $G[V \setminus (A \cup B)]$, then $A \cup S'$ is a vertex cover of $G$. 

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2. There is a minimum-cardinality vertex cover $S$ of $G$ with $A \subseteq S$.

3. For every vertex cover $S''$ of the induced subgraph $G[V \setminus (A \cup B)]$,

$$|S''| \geq \frac{|V \setminus (A \cup B)|}{2}.$$

**Problem 36.2.** Is there a generalization of Nemhauser and Trotter’s theorem for the VCP$_k$ problem?

Recently, we got a fixed-parameter algorithm with runtime $O(2^k \cdot mn^{3/2} \log n)$ for the VCP$_3$ problem using the iterative compression method. Now, we are preparing our manuscript.

**Problem 36.3.** Is there a fix-parameter algorithm for the VCP$_k$ ($k \geq 4$) problem?

**References:**


