## Contents

### Colourings

1. **On the cyclic chromatic number of 3-connected plane graphs**  
   (Mirko Horňák)  
   Page 6

2. **Packing colouring in some special classes of planar graphs**  
   (Jan Ekstein, Přemysl Holub)  
   Page 7

3. **Colouring vertices of plane graphs under restrictions given by faces**  
   (Stanislav Jendrol')  
   Page 8

4. **List colourings of integer distance graphs**  
   (Arnfried Kemnitz)  
   Page 9

5. **[1, 1, t]-Colourings of Complete Graphs**  
   (Arnfried Kemnitz, Massimiliano Marangio)  
   Page 10

6. **Polychromatic colourings of cube graphs**  
   (Heiko Harborth)  
   Page 11

7. **Saturated rainbow edge colouring of cube graphs**  
   (Heiko Harborth, Arnfried Kemnitz)  
   Page 11

8. **Edge-distinguishing index of a sum of cycles**  
   (Rafał Kalinowski, Mariusz Woźniak)  
   Page 12

9. **Neighbours distinguishing index of planar graphs**  
   (Keith Edwards, Mirko Horňák, Mariusz Woźniak)  
   Page 13

10. **Edge colourings and the number of palettes**  
    (Mirko Horňák, Rafał Kalinowski, Mariusz Woźniak)  
    Page 14

11. **The neighbour-distinguishing-index by sums in total proper colourings**  
    (Monika Pilśniak, Mariusz Woźniak)  
    Page 15
12 On the chromatic number of $2K_2$-free graphs ........................................ 16
   (Ingo Schiermeyer)

Cycles

13 Every connected locally connected graph is weakly pancyclic .............. 17
   (Zdeněk Ryjáček)

14 Dominating cycles and hamiltonian prisms ............................... 19
   (Zdeněk Ryjáček)

15 Nonpancyclic claw-free graphs with complete closure .................... 20
   (Zdeněk Ryjáček, Richard Schelp)

16 Hamiltonian neighborhood graphs ....................................... 22
   (Martin Sonntag, Hanns-Martin Teichert)

17 Spanning connectivity ............................................... 23
   (Elkin Vumar)

18 Is a $\frac{3}{2}$-tough maximal planar graph hamiltonian? ............... 24
   (Jochen Harant)

19 Toughness condition for hamiltonian multisplit graphs .................. 25
   (Hajo Broersma, Elkin Vumar)

20 Hamiltonian 3-factors ........................................... 26
   (Kenta Ozeki)

Decompositions & Partitions

21 Arbitrarily vertex-decomposable trees .................................. 27
   (Mirko Horňák, Antoni Marczyk and Mariusz Woźniak)

22 Decomposing bipartite graphs into locally irregular subgraphs .......... 29
   (Olivier Baudon, Julien Bensmail, Jakub Przybyło, Mariusz Woźniak)

23 Vertex-disjoint independent sets ..................................... 30
   (Anja Kohl)
Dominating Sets

24 Domination hypergraphs of tournaments
(Martin Sonntag, Hanns-Martin Teichert)

25 On the ratio of independent domination number and order of a cubic graph
(Michael A. Henning)

Properties of Graphs

26 Weight of graphs having a given property
(Stanislav Jendrol’)

27 Turan type numbers for edges in hypercube graphs
(Heiko Harborth, Hauke Nienborg)

28 Minimum number of crossings of pseudo diagonals in convex polygons
(Heiko Harborth)

29 Multiple crossings in drawings of complete graphs
(Heiko Harborth)

30 Crossing regular cycle drawings
(Heiko Harborth)

31 Ramsey numbers for graph drawings
(Heiko Harborth)

32 Sphere graphs or Kuratowski in R^3
(Heiko Harborth)

33 Straight ahead cycles in drawings of eulerian graphs
(Heiko Harborth)

Paths

34 Minimum k-path vertex cover
(Jan Katrenič, Ingo Schiermeyer, Gabriel Semanišin)

Notes

.......................................................... 40
Colourings

1 ON THE CYCLIC CHROMATIC NUMBER OF
3-CONNECTED PLANE GRAPHS

(Mirko Horňáčk)

The cyclic chromatic number of a plane graph G, in symbol \( \chi_c(G) \), is a minimum number of colours in such a vertex colouring of G that distinct vertices incident with a common face receive distinct colours. If G is 2-connected, then \( \chi_c(G) \geq \Delta^*(G) \), where \( \Delta^*(G) \) is the maximum face degree of G.

On the other hand, no 3-connected plane graph G is known with \( \chi_c(G) > \Delta^*(G) + 2 \).

Plummer and Toft ([7]) proved that \( \chi_c(G) \leq \Delta^*(G) + 9 \) and conjectured (PTC) that \( \chi_c(G) \leq \Delta^*(G) + 2 \) for any 3-connected plane graph G.

Let PTC\((d)\) denote PTC restricted to 3-connected plane graphs G with \( \Delta^*(G) = d \).

It is known that PTC\((d)\) is true for \( d = 3 \) (Four Colour Theorem), \( d = 4 \) (Borodin [1]), \( d \in \{18, \ldots, 23\} \) (Horňák and Zlámalová [6]) and \( d \geq 24 \) (Horňák and Jendrol’ [5]). For \( \Delta^*(G) \geq 60 \) Enomoto et al. ([4]) obtained the best possible inequality:

\[ \chi_c(G) \leq \Delta^*(G) + 1 \] (graphs of pyramids show that the bound \( \Delta^*(G) + 1 \) cannot be improved).

The best general upper bound known so far is due to Enomoto and Horňák ([3]), namely \( \chi_c(G) \leq \Delta^*(G) + 5 \).

**Problem 1.1.** Prove PTC\((d)\) for some \( d \in \{5, \ldots, 17\} \).

**References:**


[2] O. V. Borodin, D. P. Sanders, Y. Zhao, On cyclic colorings and their generalizations


2 Packing colouring in some special classes of planar graphs

(Jan Ekstein, Přemysl Holub)

Let $G$ be a connected graph and let $\text{dist}_G(u, v)$ denote the distance between vertices $u$ and $v$ in $G$. A partition of the vertex set of $G$ into (not necessarily nonempty) disjoint classes $X_1, \ldots, X_k$ such that each colour class $X_i$ is an $i$-packing is called a packing $k$-colouring. Each $i$-packing $X_i$ is a set of vertices such that any distinct pair $u, v \in X_i$ satisfies $\text{dist}_G(u, v) > i$. The smallest integer $k$ for which there exists a packing $k$-colouring of $G$ is called the packing chromatic number of $G$, denoted $\chi_\rho(G)$.

The determination of the packing chromatic number is known to be $\mathcal{NP}$-hard for general graphs [3] and Fiala and Golovach [1] showed that the problem remains $\mathcal{NP}$-hard even for trees.

Sloper proved in [4] that the packing chromatic number of a complete infinite binary tree is equal to 7 while a complete infinite ternary tree cannot be coloured using a finite number of colours. Hence we will focus on cubic graphs. Since there are planar graphs for which $\chi_\rho$ is not finite (e.g. the infinite triangular grid [3]), we can consider some special subclasses of planar graphs. Thus we formulate the following questions:

**Problem 2.1.** Is there a constant $c$ such that, if $G$ is a cubic planar graph, then $\chi_\rho(G) \leq c$?

**Problem 2.2.** Is there a constant $c$ such that, if $G$ is a outerplanar graph, then $\chi_\rho(G) \leq c$?

**References:**


3 COLOURING VERTICES OF PLANE GRAPHS UNDER RESTRICTIONS GIVEN BY FACES

(Stanislav Jendrol’)

Consider a vertex colouring of a connected plane graph $G$. A colour $c$ is used $k$ times by a face $\alpha$ of $G$ if it appears $k$ times along the facial walk of $\alpha$. Two natural problems arise.

1. A vertex colouring $\varphi$ is a weak parity vertex colouring of a connected plane graph $G$ with respect to its faces if each face of $G$ uses at least one colour an odd number of times. Problem is to determine the minimum number $\chi_w(G)$ of colours used in a wpv colouring of $G$.

In [1] it is proved that $\chi_w(G) \leq 4$ for every connected plane graph $G$ with minimum degree at least 3. We strongly believe that the following holds.

Conjecture 3.1. Let $G$ be a connected plane graph of minimum face degree at least 3. Then

$$\chi_w(G) \leq 3.$$ 

The Conjecture is true for 2-connected cubic plane graphs, see [1].

2. A vertex colouring $\varphi$ is a strong parity vertex colouring of a 2-connected plane graph $G$ with respect to the faces of $G$ if each face of $G$ that uses a colour then it uses an odd number of times. Problem is to find the minimum number $\chi_s(G)$ of colours used in an spv colouring of $G$. We believe that

Conjecture 3.2. There is a constant $k$ such that for every 2-connected plane graph $G$

$$\chi_s(G) \leq k.$$ 

We do not know any 2-connected plane graph $H$ with $\chi_s(H) \geq 7$. Hence, we believe that $k = 6$ in the above conjecture.
4 LIST COLOURINGS OF INTEGER DISTANCE GRAPHS

(Arnfried Kemnitz)

Let $D$ be a subset of the positive integers $\mathbb{N}$. The integer distance graph $G(\mathbb{Z}, D) = G(D)$ is defined as the graph with the set of integers as vertex set, $V(G(D)) = \mathbb{Z}$, and edge set consisting of all pairs $uv$ whose distance $|u - v|$ is an element of the so-called distance set $D$.

General bounds for the chromatic number of integer distance graphs are

$$2 \leq \chi(G(D)) \leq |D| + 1.$$ 

Voigt ([4]) and Zhu ([5]) determined $\chi(G(D))$ if $|D| = 3$:
If $D = \{x, y, z\}$ consists of integers whose greatest common divisor equals 1, then $\chi(D) = 4$ if and only if $D = \{1, 2, 3n\}$ or $D = \{x, y, x + y\}$ and $x \equiv y \pmod{3}$. If $x, y, z$ are odd then $\chi(D) = 2$. For all other 3−element distance sets $D$ it holds $\chi(D) = 3$.

General bounds for the list chromatic number (choice number) of integer distance graphs are $\chi(D) \leq \chi_l(D) \leq |D| + 1$ (Kemnitz, Marangio 2001).

**Question 4.1.** Does there exist a 3−element distance set such that $\chi_l(D) < 4$?

**References:**


Discussiones Mathematicae Graph Theory 22 (2002), 149-158.

Ars Combinatoria 52 (1999), 3-12.

[5] X. Zhu, *Distance graphs on the real line*  
manuscript, 1996.
5 \( [1, 1, t] \)-COLOURINGS OF COMPLETE GRAPHS

(Arnfried Kemnitz, Massimiliano Marangio)

Given non-negative integers \( r, s, \) and \( t, \) an \([r, s, t]\)-colouring of a graph \( G = (V(G), E(G)) \) is a mapping \( c \) from \( V(G) \cup E(G) \) to the colour set \( \{0, 1, \ldots, k - 1\} \) such that \( |c(v) - c(v')| \geq r \) for every two adjacent vertices \( v, v' \), \( |c(e) - c(e')| \geq s \) for every two adjacent edges \( e, e' \), and \( |c(v) - c(e)| \geq t \) for all pairs of incident vertices and edges, respectively. The \([r, s, t]\)-chromatic number \( \chi_{r,s,t}(G) \) of \( G \) is defined to be the minimum \( k \) such that \( G \) admits an \([r, s, t]\)-colouring.

This is an obvious generalization of all classical graph colourings since \( c \) is a vertex colouring if \( r = 1, s = t = 0 \), an edge colouring if \( s = 1, r = t = 0 \), and a total colouring if \( r = s = t = 1 \), respectively. Therefore, \( \chi_{1,0,0}(G) = \chi(G), \chi_{0,1,0}(G) = \chi'(G), \) and \( \chi_{1,1,1}(G) = \chi''(G) \) where \( \chi(G) \) is the chromatic number, \( \chi'(G) \) the chromatic index, and \( \chi''(G) \) the total chromatic number of the graph \( G \).

For complete graphs \( K_n \) on \( n \) vertices it holds

\[
\chi_{1,1,1}(K_n) = \chi''(K_n) = \begin{cases} n & \text{if } n \text{ odd}, \\ n + 1 & \text{if } n \text{ even} \end{cases}
\]

and we proved (see [1] and [2])

\[
\chi_{1,1,2}(K_n) = \begin{cases} n & \text{if } n = 1, \\ n + 2 & \text{if } n \geq 3 \text{ odd, } n = 2, \ n = 6, \text{ or } n = 8, \\ n + 3 & \text{if } n = 4 \text{ or } n \geq 10 \text{ even,} \end{cases}
\]

\[
\chi_{1,1,t}(K_n) = 2n + t - 2 \text{ for } n \geq 2 \text{ and } t \geq n, \\
\chi_{1,1,t}(K_n) = n + 2t - 2 \text{ for } n \geq 3 \text{ and } t \geq 3 \text{ with } \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq t \leq n - 1.
\]

**Problem 5.1.** Is it true that \( \chi_{1,1,t}(K_n) = n + 2t + u \) if \( t < n \) where \( u \) is a small (positive or negative) constant (depending on \( n \) and \( t \))?

**Problem 5.2.** Determine \( \chi_{1,1,t}(K_n) \) for \( 3 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 \).

**References:**


6  POLychromatic colourINGS OF CUBe GRaphs

(Heiko Harborth)

The polychromatic numbers $e(n,k)$ and $g(n,k)$ denote the maximum numbers of colours for the edges and for the vertices, respectively, of the $n$-dimensional cube graph such that each $k$-dimensional subcube graph contains all colours. See [1] for the limits of $e(n,k)$ and $g(n,k)$ if $k$ is fixed. - What about exact values of $e(n,k)$ and $g(n,k)$? See [2] for some first exact values of $g(n,k)$.

References:


7  SATURATED RAINBOW EDGE COLOURING OF CUBE GRAPHS

(Heiko Harborth, Arnfried Kemnitz)

Let $f(n, k)$ denote the minimum number of colours for the edges of the cube graph $Q_n$ such that for $k < n$ no rainbow $Q_k$ occurs (the edges of a rainbow $Q_k$ have pairwise different colours), however, for every edge with a colour used at least twice it follows that a new colour for this edge induces a rainbow $Q_k$.

For $k = 3$ it is known $f(3, 3) = 11$, $f(4, 3) = 22$, and $f(5, 3) = 20$.

1. Determine $f(5, 3)$ and $f(6, 3)$.

2. Determine the smallest $n > 3$ such that $f(n, 3) = 11$.

References:

8 Edge-distinguishing index of a sum of cycles

(Rafał Kalinowski, Mariusz Woźniak)

A neighbourhood $N(e)$ of an edge $e$ of a graph $G = (V, E)$ is a subgraph of $G$ induced by $e$ and all edges adjacent to $e$. A colouring $c : E \to S$ is called edge-distinguishing if, for any two distinct edges $e, e'$, there does not exist an isomorphism $\varphi$ of $N(e)$ onto $N(e')$ preserving colours of $c$, and such that $\varphi(e) = e'$. An edge-distinguishing index $\chi_e'(G)$ of a graph $G$ is the minimum number of colours in a proper edge-distinguishing colouring $c : E \to S$.

If $G$ is a cycle $C_n$ of length $n$, then it is easy to see that

$$\chi_e'(C_n) \geq \gamma_n := \min\{ k \mid \frac{1}{2} k^2(k-1) \geq n \}.$$

**Theorem 8.1.**

$$\chi_e'(C_n) = \begin{cases} \gamma_n + 1 & \text{if } n = \frac{1}{2} k^2(k-1) - 1 \text{ or } n = 4, \\ \gamma_n & \text{otherwise.} \end{cases}$$

**Problem 8.1.** Let $G$ be a disjoint sum of cycles with the total sum of lengths equal to $n$. Evaluate $\chi_e'(G)$. 
9 Neighbours distinguishing index of planar graphs

(Keith Edwards, Mirko Horňák, Mariusz Woźniak)

Let $G$ be a finite simple graph with no component $K_2$. Let $C$ be a finite set of colours and let $\varphi : E(G) \to C$ be a proper edge colouring of $G$. The colour set of a vertex $v \in V(G)$ with respect to $\varphi$, in symbols $S_\varphi(v)$, is the set of colours of edges incident with $v$. The colouring $\varphi$ is neighbours distinguishing if $S_\varphi(x) \neq S_\varphi(y)$ for any $xy \in E(G)$. For example, any neighbour-distinguishing colouring of $C_5$ uses necessarily 5 colours.

The neighbours distinguishing index of the graph $G$ is the smallest number $\text{ndi}(G)$ of colours in a neighbour-distinguishing colouring of $G$. Neighbours distinguishing index has been introduced in [7], where the authors have conjectured that $\text{ndi}(G) \leq \Delta(G) + 2$ for any connected graph $G$ nonisomorphic to $C_5$ on at least three vertices (Neighbour-Distinguishing Conjecture = NDC). NDC was confirmed in [1] for cubic graphs and for bipartite graphs, in [6] for graphs with maximum degree at most 3, in [2] for planar graphs with girth at least 6 and in [5] for planar graphs with maximum degree at least 12. In [3] it was proved that $\text{ndi}(G) \leq \Delta(G) + 1$ for any planar bipartite graph $G$ with $\Delta(G) \geq 12$. Hatami in [4] showed that $\text{ndi}(G) \leq \Delta(G) + 300$ provided that $\Delta(G) > 10^{20}$.

**Problem 9.1.** Find the minimum integer $\Delta \geq 4$ such that $\text{ndi}(G) \leq \Delta(G) + 1$ for any plane bipartite graph $G$ with $\Delta(G) \geq \Delta$.

**Problem 9.2.** Prove or disprove NDC for planar graphs $G$ with $\Delta(G) = \Delta$ for (at least some) $\Delta \in \{4, \ldots, 11\}$.

**References:**


10 **EDGE COLOURINGS AND THE NUMBER OF PALETTES**

(Mirko Horňák, Rafał Kalinowski, Mariusz Woźniak)

Let $G$ be a finite simple graph of order $n$. Let $C$ be a finite set of colours and let $\varphi : E(G) \rightarrow C$ be a proper edge colouring of $G$. The palette of a vertex $v \in V(G)$ (with respect to $\varphi$) is the set of colours of edges incident with $v$. The number of (distinct) palettes ranges between 1 and $n$ and any colouring that uses $|E(G)|$ colours produces the maximum number $n$ of palettes (except for graphs having two isolated vertices or containing $K_2$ as a component). So, we are interested in the minimum number of palettes taken over all possible proper colourings of $G$. We call this number the *palette index* of $G$ and denote it by $s(G)$.

**Known results:** We have proved in [1]:

- $s(G) = 1 \Leftrightarrow G$ regular, class 1
- $n \equiv 3 \pmod{4} \Rightarrow s(K_n) = 3$
- $n \equiv 1 \pmod{4} \Rightarrow s(K_n) = 4$
- $G$ connected, cubic, class 2, having a perfect matching $\Rightarrow s(G) = 3$
- $G$ connected, cubic, class 2, having no perfect matching $\Rightarrow s(G) = 4$

Thus Rathen Problem 14.1/2011 has been completely solved, the sequence $\{s(K_n)\}_{n=1}^{\infty}$ is bounded.

**Problem 10.1.** Determine $s(K_p \times q)$ for the complete $p$-partite graph with all parts of cardinality $q$.

**References:**

11 THE NEIGHBOUR-DISTINGUISHING-INDEX BY SUMS IN TOTAL PROPER COLOURINGS

(Monica Pilśniak, Mariusz Woźniak)

Let \( c : V \cup E \rightarrow \{1, 2, \ldots, k\} \) be a proper total coloring of a graph \( G \). For a vertex \( v \), we denote by \( f(v) \) the sum of colors of the edges incident to \( v \) and of the color of \( v \). The smallest \( k \) that guarantees that there is a coloring \( c \), so that the function \( f \) distinguishes adjacent vertices of \( G \), is called the total-neighbour-distinguishing-index by sums of \( G \), and it is denoted by \( tndi_{\Sigma}(G) \).

If we consider either trees and regular bipartite graphs, then \( tndi_{\Sigma}(G) \) equals to \( \Delta + 1 \), if there does not exist two adjacent vertices of maximum degree, or \( tndi_{\Sigma}(G) = \Delta + 2 \), otherwise.

For complete graphs, we can show that the total-neighbour-distinguishing-index depends on the parity of their order. Namely,

\[
tndi_{\Sigma}(K_n) = \begin{cases} 
  n + 1, & \text{if } n \text{ is even}, \\
  n + 2, & \text{if } n \text{ is odd}. 
\end{cases}
\]

We can also show that, if \( G \) is a bipartite graph, or a cubic graph, or a graph with \( \Delta \leq 3 \), then \( tndi_{\Sigma}(G) \leq \Delta + 3 \). Hence, we may conjecture:

**Conjecture 11.1.** ([2]) For every graph \( G = (V, E) \), the total-neighbour-distinguishing-index by sums satisfies the inequality

\[
tndi_{\Sigma}(G) \leq \Delta + 3.
\]

In [3], the authors investigated also a proper total coloring of \( G \), but for every vertex \( v \) they assigned a set \( S(v) \) of colors of the edges incident to \( v \) and the color of \( v \). Similarly as above, by \( tndi(G) \) we can denote the smallest number \( k \) of colors, so that there exists a proper total coloring \( c \) for \( G \) and \( S(u) \) is different from \( S(v) \) for every pair of adjacent vertices \( u, v \).

**Conjecture 11.2.** ([3]) For every graph \( G = (V, E) \), the total-neighbour-distinguishing-index by sets \( tndi(G) \) satisfies the inequality

\[
tndi(G) \leq \Delta + 3.
\]

Zhang, Chen, Li, Yao, Lu and Wang considered the cases of cliques, paths, cycles, fans, wheels, stars, complete graphs, bipartite complete graphs and trees. They showed that \( \Delta + 3 \) colors are enough in these cases. Next in [1] Chen proved this conjecture for bipartite graphs and for graphs with maximum degree at most three.

It is easy to observe, that if two vertices are distinguished by sums then they are also distinguished by sets, but not necessarily conversely.
On the chromatic number of $2K_2$-free graphs

(Ingo Schiermeyer)

In this problem we consider graphs without induced subgraphs $2K_2$. In 1980 Wagon has shown the following upper bound for the chromatic number of a $2K_2$-free graph $G$.

**Theorem 12.1.** [2]
Let $G$ be a $2K_2$-free graph with clique number $\omega(G)$. Then $\chi(G) \leq \left(\frac{\omega(G) + 1}{2}\right)$.

Gyárfás [1] has posed the following problem (as Problem 2.16 in [1]). Let $\mathcal{G}(2K_2)$ denote the class of all $2K_2$-free graphs.

**Problem 12.2.** What is the order of magnitude of the smallest $\chi$-binding function for the class $\mathcal{G}(2K_2)$?

A lower bound is $\frac{R(C_4, K_{\omega+1}) - 1}{3}$, where $R(C_4, K_{\omega+1})$ denotes the Ramsey number, i.e. the smallest $n$ such that every graph on $n$ vertices contains either a clique of size $\omega + 1$ or the complement of the graph contains a $C_4$ (a cycle on four vertices).

Concerning particular values of the smallest $\chi$-binding function $f^*$ for $\mathcal{G}(2K_2)$ it has been shown that $f^*(2) = 3$ and $f^*(3) = 4$.

For subclasses of $\mathcal{G}(2K_2)$ the following results are known.

1. If $G$ is a $(2K_2, C_4)$-free graph, then $\chi(G) \leq \omega(G) + 1$ (Blazsik, Hujter, Pluhar, and Tuza, 1993).

2. If $G$ is a $(2K_2, P_5)$-free graph, then $\chi(G) \leq \frac{3}{2}\omega(G)$ (Schiermeyer, 2016). This bound is best possible and $C_4 \subset P_5$.

References:


Cycles

13 EVERY CONNECTED LOCALLY CONNECTED GRAPH IS WEAKLY PANCYCLIC

(Zdeněk Ryjáček)

Let $G$ be a finite simple undirected graph and let $g(G)$ and $c(G)$ be the girth and the circumference of $G$ (i.e. the length of a shortest cycle of $G$ and the length of a longest cycle of $G$), respectively. We say that $G$ is weakly pancyclic if $G$ contains cycles of all lengths $\ell$ for $g(G) \leq \ell \leq c(G)$. The graph $G$ is locally connected if the neighborhood of every vertex of $G$ induces a connected graph.

Conjecture 13.1. ([8]) Every connected locally connected graph is weakly pancyclic.

Comments: The concept of locally connected graphs was introduced by Chartrand and Pippert ([3]). More information about weakly pancyclic graphs appears in [2], for example.

The conjecture is based on a result by Clark ([4]), who proved that every connected, locally connected graph is vertex pancyclic (having cycles of all lengths from 3 to $|V(G)|$ through every vertex). Without the claw-free assumption, it is easy to construct locally connected graphs that are nonhamiltonian. Nevertheless, all known examples are weakly pancyclic; and indeed [4] proved the conjecture for claw-free graphs.

In a chordal graph, every block is locally connected, and for every cycle of length at least 4 there is a cycle with length one less that is obtained by skipping one vertex. Thus the conjecture holds for chordal graphs.

It is easy to show that the square of any graph is locally connected. (The square adds edges making vertices at distance 2 in the original graph adjacent.) Fleischner ([5], Theorem 6) proved that the square of every graph is weakly pancyclic, thus...
verifying the conjecture for squares of graphs.

The lexicographical product of graphs is another way to obtain a locally connected graph. Kaiser and Kriesell ([6]) recently proved that the lexicographical product $G[H]$ is weakly pancyclic provided $G$ is a connected graph and $H$ is an arbitrary graph with at least one edge.

Kriesell ([7]) verified the conjecture for graphs with maximum degree at most 4.

Finally, planar triangulations are locally connected. Balister ([1]) proved the conjecture for this class as follows. Let $C$ be a cycle in a planar triangulation $G$. By induction on the number of faces inside, we prove that the interior (with boundary) contains cycles of all shorter lengths. If some face inside has two edges on $C$, then using the third edge yields a cycle $C'$ with length one less and fewer faces inside. Otherwise, there is a face with one edge on $C$ and the third vertex inside. Detouring from $C$ to include this vertex forms a longer cycle $C'$, but again it has fewer regions inside and the induction hypothesis applies.

References:

   J. Graph Theory 27 (1998), 141-176.
[5] H. Fleischner, In the square of graphs, hamiltonicity and pancyclicity, hamiltonian connectedness and pancon-
    nectedness are equivalent concepts.
   Monatshefte für Mathematik 82 (1976), 125-149.
   Graphs Comb. 22 (2006), 51-58.
14 DOMINATING CYCLES AND HAMILTONIAN PRISMS

(Zdeněk Ryjáček)

The prism over a graph \( G \), denoted \( G \Box K_2 \), is the Cartesian product of \( G \) and \( K_2 \). It consists of two disjoint copies of \( G \) and a perfect matching connecting a vertex in one copy of \( G \) to its “clone” in the other copy.

A graph \( G \) is hamiltonian if it has a hamiltonian cycle and traceable if it has a hamiltonian path. Define a \( k \)-walk in a graph to be a spanning closed walk in which every vertex is visited at most \( k \) times.

The following implications are easy to verify:

\[
G \text{ is hamiltonian } \implies G \text{ is traceable } \implies G \Box K_2 \text{ is hamiltonian } \\
\implies G \text{ has a 2-walk.}
\]

Thus the question whether \( G \Box K_2 \) is hamiltonian is “sandwiched” between hamiltonicity and having a 2-walk. Specifically, the property of having a hamiltonian prism can be considered as a “relaxation” of hamiltonicity. More information about prism-hamiltonicity of a graph can be found e.g. in [1] and [2].

A dominating cycle in a graph \( G \) is a cycle \( C \) such that every edge of \( G \) has at least one vertex on \( C \), i.e. such that the graph \( G - C \) is edgeless. Clearly, a hamiltonian cycle is dominating, and hence the property of having a dominating cycle can be considered as another relaxation of hamiltonicity.

There is a natural question whether there is any relation between these two properties.

Example 14.1. Let \( H \) be any 2-connected cubic nonhamiltonian graph, and let \( G \) be obtained from \( H \) by replacing every vertex of \( H \) with a triangle (such a \( G \) is sometimes called the inflation of \( H \)). Then \( G \) is a 2-connected line graph and these are known [2] to be prism-hamiltonian. On the other hand, since \( H \) is nonhamiltonian, any cycle in \( G \) has to miss at least one “new” triangle and hence \( G \) has no dominating cycle. Thus, there are “many” graphs showing that hamiltonian prism does not imply having a dominating cycle.

Example 14.2. The graph in the figure below shows that also the existence of a dominating cycle does not imply having hamiltonian prism.
However, all such known examples are of low toughness. This motivates the following question.

**Conjecture 14.1.** Let $G$ be a 1-tough graph having a dominating cycle. Then $G$ has hamiltonian prism.

**Comments:** Recall that $G$ is 1-tough if, for any $S \subset V(G)$, the graph $G - S$ has at most $|S|$ components.

Suppose that $G$ has a dominating cycle $C$ of even length. Set $M = V(G) \setminus V(C)$ and $N = \{ x \in V(C) | x \text{ has a neighbor in } M \}$. Then the graph induced by $M \cup N$ has a matching containing all vertices from $M$ (this follows by the toughness assumption and by the Hall’s theorem). Using this matching, it is easy to construct a hamiltonian cycle in $G \Box K_2$.

The difficult case is when all dominating cycles in $G$ are of odd length.

**References:**


---

15 **Nonpancylic claw-free graphs with complete closure**

(Zdeněk Ryjáček, Richard Schelp)

It is known that a claw-free graph $G$ is hamiltonian if and only if its closure $\text{cl}(G)$ is hamiltonian. On the other hand, there are nonpancyclic graphs with pancyclic closure [1]. The graph in the figure below is an example of such a nonpancyclic graph with complete (and hence pancyclic) closure.
Problem 15.1. Determine the maximum number of cycle lengths that can be missing in a claw-free graph on \( n \) vertices with complete closure.

It is easy to see that a claw-free graph with complete closure on at least 4 vertices can miss neither a \( C_3 \) nor a \( C_4 \). The main result of [2] shows that such a graph \( G \) cannot be missing a cycle of length \( n - 1 \); however, the proof of this result is difficult and cannot be iterated.

The following was conjectured in [2].

Conjecture 15.1. Let \( c_1, c_2 \) be fixed constants. Then for large \( n \), any claw-free graph \( G \) of order \( n \) whose closure is complete contains cycles \( C_i \) for all \( i \), where \( 3 \leq i \leq c_1 \) and \( n - c_2 \leq i \leq n \).

Recently, counterexamples to the first part of the Conjecture 15.1 have been found (see [3]), all these counterexamples have connectivity \( \kappa \leq 5 \). We believe that the second part of Conjecture 15.1 is true, and that such a construction as shown in [3] is possible only for connectivity \( \kappa \leq 5 \). Thus, we conjecture the following.

Conjecture 15.2. Let \( c \) be a fixed constant. Then for large \( n \), any claw-free graph \( G \) of order \( n \), whose closure is complete, contains cycles \( C_i \) for all \( i \), \( n - c \leq i \leq n \).

Conjecture 15.3. Every 6-connected claw-free graph with complete closure is pancyclic.

References:


16 Hamiltonian neighborhood graphs

(Martin Sonntag, Hanns-Martin Teichert)

The neighborhood graph $N(G)$ for a simple graph $G = (V, E)$ is defined to be the graph on the same vertex set $V$ with two vertices adjacent if and only if there is in $G$ a path of length two between them. Neighborhood graphs, also referred to as two-step graphs, have been the object of several studies in the last 25 years.

If $G$ is hamiltonian then $N(G)$ is hamiltonian for $|V|$ odd. This is not true for $|V|$ even, for instance $N(C_{2n})$ is disconnected. We can show that $N(G)$ is always hamiltonian if $G$ is 1-hamiltonian connected and has a triangle, but we think there are weaker conditions providing hamiltonicity of $N(G)$.

**Problem 16.1.** Find sufficient conditions for $G$, such that $N(G)$ is hamiltonian or hamiltonian connected.
17 **SPANNING CONNECTIVITY**

(Elkin Vumar)

Let $G = (V, E)$ be a connected graph. For $u, v \in V(G)$, a $k^*$-container $C(u, v)$ is the set of $k$ internally disjoint $(u, v)$-paths that contains all vertices of $G$. $G$ is $k^*$-connected if there is a $k^*$-container between any distinct pair of vertices. By definition, $G$ is $1^*$-connected if and only if it is Hamilton-connected, and $G$ is $2^*$-connected if and only if it is Hamiltonian.

**Problem 17.1.** Determine the minimum number $f = f(k, \kappa, d)$ such that the $f$-th power $G^f$ of a graph $G$ with connectivity $\kappa \geq 1$ and diameter $d$ is $k^*$-connected. In particular, prove or disprove that $G^2$ is $3^*$-connected for a 2-connected graph $G$.

**Some known results.**

1. $C_n^2$ is $3^*$-connected for an $n$ cycle $C_n (n \geq 3)$ [1]. Hence $G^2$ is $3^*$-connected for a hamiltonian graph $G$.

2. If $G$ is a connected graph with $|V(G)| \geq k + 1 \geq 4$, then $G^k$ is $k^*$-connected [2].

**References:**


18 Is a $\frac{3}{2}$-TOUGH MAXIMAL PLANAR GRAPH HAMILTONIAN?

(Jochen Harant)

Let $G$ be a connected graph. A subset $S$ of $V(G)$ separates $G$ if the graph $G - S$ obtained from $G$ by deleting the vertices of $S$ is disconnected. $G$ is said to be $k$-tough if for every separating set $S \subseteq V(G)$ the number of components of $G - S$ is at most $|S|/k$. The toughness $\tau(G)$ of a non-complete graph $G$ is defined to be the largest $k > 0$ such that $G$ is $k$-tough ([1]). Obviously, if $\tau(G) > \frac{3}{2}$ for a graph $G$, then $G$ is 4-connected. Since a 4-connected planar graph is hamiltonian ([4]), Corollary 20.1 follows.

**Corollary 18.1.** If $\tau(G) > \frac{3}{2}$ for a planar graph $G$, then $G$ is hamiltonian.

In [2], infinitely many non-hamiltonian planar graphs $G$ with $\tau(G) = \frac{3}{2}$ are presented, however, none of them is maximal planar.

For maximal planar graphs, the following Theorem 18.2 is proved in [3].

**Theorem 18.2 ([3]).** For arbitrary $\varepsilon > 0$, there is a non-hamiltonian maximal planar graph with toughness greater than $\frac{3}{2} - \varepsilon$.

It remains open whether there exists a non-hamiltonian $\frac{3}{2}$-tough maximal planar graph.

**References:**


19 Toughtness condition for hamiltonian multisplit graphs

(Hajo Broersma, Elkin Vumar)

A graph is called a split graph if its vertex set can be partitioned into a clique and an independent set; alternatively a split graph can be viewed as the intersection graph of a family of connected subgraphs of a star (and so split graphs are chord graphs). A graph $G = (V, E)$ is called a $k$-multisplit graph ($k \geq 2$) if $V$ can be partitioned into two disjoint sets $S$ and $T$ such that $S$ is an independent set in $G$ and $T$ induces a complete $k$-partite subgraph in $G$. For $k = 2$, the class coincides with the class of bisplit graphs [1], and in the degenerated case that each class in the $k$-partition consists of only one vertex, the $k$-multisplit graph is just a split graph. The class of multisplit graphs is an extension of the class of split graphs.

Problem. Is there a real number $t$ with $3/2 \leq t < k$ such that every $t$-tough $k$-multisplit graph on at least three vertices is hamiltonian. Note that it is shown that every $k$-tough $k$-multisplit graph on at least three vertices is hamiltonian [7].

Some known results.

1. Every $3/2$-tough split graph on at least three vertices is hamiltonian, and this is the best possible in the sense that there is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor and $\tau(G_n) \to 3/2$ (see [2]). There are some other extensions along this line (see [3]-[6]).

2. Every $k$-tough $k$-multisplit graph on at least three vertices is hamiltonian [7].

3. For any fixed integer $k \geq 2$, the toughness of a $k$-multisplit graph can be determined in polynomial time [7].

References:

20  Hamiltonian 3-factors

(Kenta Ozeki)

For plane graphs with high connectivity, it is known that several good properties hold; A spanning 3-regular graph is called a 3-factor.

**Theorem 20.1** (Tutte [2]). Every 4-connected plane graph is Hamiltonian.

**Theorem 20.2** (Kawarabayashi and Ozeki [1]). Every 5-connected plane graph has a 3-factor.

We expect that the conclusion of both Theorems 20.1 and 20.2 hold “simultaneously”.

**Problem 20.3.** Is it true that every 5-connected plane graph has a Hamiltonian 3-factor, i.e. a 3-factor containing a Hamiltonian cycle?

Or, if it is too strong, what about the following weaker versions?

- Can the stronger assumption “5-connected plane triangulation” guarantee the existence of a Hamiltonian 3-factor?

- Instead of a Hamiltonian 3-factor, can we find “a 3-edge-colorable 3-factor”, “a 3-factor containing a 2-factor”, or “a 2-edge-connected 3-factor”?

(Note: It is well-known that every 2-edge-connected cubic graph contains a 2-factor. 4 Color Theorem implies that a 2-edge-connected cubic plane graph is always 3-edge-colorable. Therefore, the last one implies both of the first two.)

Those questions comes from the discussion with Shoichi Tsuchiya (Senshu University) and Michitaka Furuya (Kitasato University).

**References:**


21 Arbitrarily vertex-decomposable trees

(Mirko Horňák, Antoni Marczyk and Mariusz Woźniak)

A tree T is said to be arbitrarily vertex decomposable if for any sequence \((t_1, \ldots, t_k)\) of positive integers adding up to \(|V(T)|\) there is a sequence \((T_1, \ldots, T_k)\) of vertex-disjoint subtrees of T such that \(|V(T_i)| = t_i\) for \(i = 1, \ldots, k\).

The notion of an arbitrarily vertex decomposable (avd for short) tree has been introduced independently by Barth et al. in [1] and Horňák and Woźniak in [5].

It turned out that some classes are essential when analysing the property of a tree “to be avd”. A star-like tree (a spider) is a tree homeomorphic to a star \(K_{1,q}\). Such a tree is uniquely (up to isomorphism) determined by the non-decreasing sequence \((a_1, \ldots, a_q)\) of orders of its arms; it will be denoted by \(S(a_1, \ldots, a_q)\) and also called a q-spider.

A caterpillar is a tree T having as a subgraph a path P such that \(T - P\) is an edgeless graph.

The most general result concerns the best upper bound \(\text{avd}_{\text{max}}\) on the maximum degree of an avd tree. In [5] it has been conjectured that \(\text{avd}_{\text{max}} \leq 6\) and conjectured that \(\text{avd}_{\text{max}} = 4\). Later it was shown that \(\text{avd}_{\text{max}} \leq 5\) ([7]) and \(\text{avd}_{\text{max}} \leq 4\) ([2]). More precisely, the result of [2] reads as follows:

**Theorem 21.1.** If a tree T is avd, then \(\Delta(T) \leq 4\). Moreover, if a tree T is avd, then each vertex of T of degree four is adjacent to a leaf.

Let us mention that there are avd trees with maximum degree 4, for example \(S(2, 2, 5, 7)\), hence, as conjectured, \(\text{avd}_{\text{max}} = 4\).

There are only few known families of avd trees. The following theorem has been proved independently in [1] and [5] (see also [4] for another, much more complicated result of this type).

**Theorem 21.2.** A 3-spider \(S(2, a_2, a_3)\) is avd if and only if \(a_2\) and \(a_3\) are coprime.

For \(a_1 \geq 2\) let \(A_2(a_1)\) be the set of all \(a_2\)’s such that \(a_2 \geq a_1\) and there is \(a_3 \geq a_2\) such that \(S(a_1, a_2, a_3)\) is avd. Similarly, for \(a_2 \geq a_1 \geq 2\) let \(A_3(a_1, a_2)\) be the set of all \(a_3\)’s such that \(a_3 \geq a_2\) and there is \(a_4 \geq a_3\) such that \(S(a_1, a_2, a_3, a_4)\) is avd.

From Theorem 21.1 it is clear that if \(A_3(a_1, a_2) \neq \emptyset\), then \(a_1 = 2\).

We give below four “main” open questions concerning avd trees.

**Question 21.1.** Is \(A_2(a_1) \neq \emptyset\) for all \(a_1 \geq 2\)?

**Question 21.2.** Is \(A_3(2, a_2) \neq \emptyset\) for all \(a_2 \geq 2\)?
Horňák and Woźniak ([6]) showed that $A_2(\alpha_1) \neq \emptyset$ for all $\alpha_1 \in \{2, \ldots, 28\}$ and $A_3(2, \alpha_2) \neq \emptyset$ for all $\alpha_2 \in \{2, \ldots, 23\}$. According to [3] there are infinitely many $\alpha_1$'s such that $A_2(\alpha_1) \neq \emptyset$.

It is easy to see that an avd caterpillar has at most one vertex of degree four. In Figure 1 there is depicted an avd tree having two vertices of degree four.

An edge labelled $l$ stands for a subpath with $l$ vertices of degree 2.

References:

[1] D. Barth, O. Baudon and J. Puech, Decomposable trees: a polynomial algorithm for tripodes


Discrete Math. 309 (2009), 3882-3887.


[5] M. Horňák and M. Woźniak, Arbitrarily vertex decomposable trees are of maximum degree at most six
Opuscula Math. 23 (2003), 49-62.

Discrete Math. 308 (2008), 1269-1281.

Decomposing bipartite graphs into locally irregular subgraphs

(Olivier Baudon, Julien Bensmail, Jakub Przybyło, Mariusz Woźniak)

A locally irregular graph is a graph whose adjacent vertices have distinct degrees. We say that a graph $G$ can be decomposed into $k$ locally irregular subgraphs if its edge set may be partitioned into $k$ subsets each of which induces a locally irregular subgraph in $G$. This is equivalent to painting the edges of $G$ with $k$ colours so that every colour class induces a locally irregular subgraph in $G$. Such property of a $k$-edge colouring is stronger than the one investigated by Addario-Berry et al. in [1], who required the neighbours in $G$ to be incident with distinct multisets of colours. This problem is also related to the ‘1-2-3 Conjecture’, see e.g. [3].

It is known that all connected graphs except for odd paths and cycles and a special family of graphs of maximum degree 3 can be decomposed into (a certain number of) locally irregular subgraphs, and it has been conjectured that each of these can be decomposed into (at most) three such subgraphs, see [2]. This has been verified e.g. for trees, complete graphs, complete bipartite graphs and for graphs of sufficiently large minimum degree, see [2] and [4].

Apart from solving the main conjecture entirely, the following problem and its weaker counterpart below seem to be intriguing.

**Conjecture 22.1.** Every connected bipartite graph which is not an odd length path can be decomposed into three locally irregular subgraphs.

**Conjecture 22.2.** There exists a constant $K$ such that every connected bipartite graph which is not an odd length path can be decomposed into (at most) $K$ locally irregular subgraphs.

Finally, one might also consider a similar weaker correspondent of the main conjecture itself.

**References:**


23 VERTEX-DISJOINT INDEPENDENT SETS

(Anja Kohl)

While investigating the minimum order of $k$–chromatic $K_{r+1}$–free graphs the following question concerning Ramsey numbers arises:

**Question 23.1.** Let $G$ be a graph with clique number at most $r$, independence number 3, and order $n = R(r + 1, 3) + 1$. Do there always exist two vertex-disjoint independent sets with 3 vertices?

**Known:**

- If the order of $G$ is $n = R(r + 1, 3) + 2$, then the answer is "yes" ([1]).
- If $G$ is a graph on at least 10 vertices then either $G$ contains a clique or an independent set on four vertices, or $G$ contains two disjoint triangles ([2]). This affirms the question for $r = 3$.

**References:**


Dominating Sets

24 DOMINATION HYPERGRAPHS OF TOURNAMENTS

(Martin Sonntag, Hanns-Martin Teichert)

Let $D = (V, A)$ be a digraph. A subset $V' \subseteq V$ is called a dominating set iff $\forall x \in V \setminus V' \exists y \in V' : (y, x) \in A$. The domination graph $\mathcal{D}(D)$ has vertex set $V$ and its edges are the dominating sets of cardinality two (see for instance [1]). As a natural generalization the domination hypergraph $\mathcal{DH}(D)$ also has vertex set $V$ and its edges are all minimal dominating sets $V'$ with $|V'| \geq 1$.

There are many interesting results on domination graphs of tournaments $T_n$, e.g. in general $\mathcal{D}(T_n)$ is not connected (see for instance [2]).

Conjecture 24.1. The domination hypergraph $\mathcal{DH}(T_n)$ of a tournament $T_n$ consists of at most one nontrivial connected component.

The conjecture is true for $n \leq 9$. We tested hundreds of bigger examples (up to $n = 23$) by Mathematica routines and found no counterexample. It is easy to prove that every nontrivial component of $\mathcal{DH}(T_n)$ contains at least three edges. A first step to verify the conjecture could be the investigation of regular tournaments.

References:


25 On the ratio of independent domination number and order of a cubic graph

(Michael A. Henning)

Problem 25.1. Prove or disprove the conjecture (see [1]) that the independent domination number of a connected cubic graph different from $K_{3,3}$ and the 5-prism $C_5 \Box K_2$ is at most three-eights its order; that is, if $G \notin \{K_{3,3}, C_5 \Box K_2\}$ is a connected cubic graph of order $n$, then $i(G) \leq 3n/8$.

What is known: The conjecture has been proven if $G$ does not have a subgraph isomorphic to $K_{2,3}$.

References:
Properties of Graphs

26 Weight of graphs having a given property

(Stanislav Jendrol’)

Let $G$ be a graph of positive size and let $e = xy$ be an edge of $G$. The weight of $e$ is $w(e) := \deg_G(x) + \deg_G(y)$ and the weight of $G$ is $w(G) := \min(w(e) : e \in E(G))$. Let $n, m \in \mathbb{Z}$, $n \geq 2$, $1 \leq m \leq \binom{n}{2}$, let $P$ be a graph property and let $P(n, m)$ be the set of all graphs in $P$ of order $n$ and size $m$. If $P(n, m) = \emptyset$, we define $w(n, m, P) := \max(w(G) : G \in P(n, m))$. In the case $P = I$ (the most general property “to be a graph”) the problem of determining $w(n, m, I)$ was formulated by Erdős during the conference in Prachatice held in 1990 and solved by Jendrol’ and Schiermeyer in [1].

We have obtained the results mentioned below.

Let $B$ denote bipartite graphs. For a pair $(n, m)$ such that $B(n, m) \neq \emptyset$ let $a, b, s, p, w$ be integers defined by $a := \lceil n - \sqrt{n^2 - 4m} \rceil$, $b := \lceil m/a \rceil$, $s := ab - m$, $p := \min(s, 2)$ and $w := a + b - p$.

**Theorem 26.1.** If $n \geq 2$ and $1 \leq m \leq \lceil n^2/2 \rceil$, then the following hold:

1. $1 \leq w \leq w(n, m, B) \leq w + 1 \leq n$.
2. If $w(n, m, B) = w + 1$, then $w(n, m, B) = n$.
3. $w(n, m, B) = n$ if and only if $\sqrt{n^2 - 4m}$ is an integer.
4. $w(n, m, B) = n - 1$ if and only if one of the numbers $\sqrt{n^2 - 4m - 4}$ and $\sqrt{(n - 1)^2 - 4m}$ is an integer.
5. If $\sqrt{n^2 - 4m}$, $\sqrt{n^2 - 4m - 4}$ and $\sqrt{(n - 1)^2 - 4m}$ are not integers, then $w(n, m, B) = w$.

Let $C$ denote connected graphs and $D_{1+}$ graphs with minimum degree at least 1. For a pair $(n, m)$ such that $C(n, m) \neq \emptyset$ let $k, m', r, c, d, e$ be integers defined by
\[
\binom{n}{2} - \frac{(k+1)}{2} < m \leq \binom{n}{2} - \frac{k}{2}, \quad m' := \binom{n}{2} - \frac{k}{2} - m, \quad r := \left\lfloor \frac{m'}{n-k} \right\rfloor, \quad c := 1 \text{ if either } m' \leq \left\lfloor \frac{n-k}{2} \right\rfloor \text{ or } m' = (n - k - 1)^2, \quad c := 2 \text{ otherwise}, \quad d := \left\lfloor \frac{2m}{n} \right\rfloor, \quad e := 0 \text{ if either } m = \binom{n}{2} - 1 \text{ or } d \leq n - 3 \text{ and } 2m \equiv q \pmod{n}, \quad 0 \leq q \leq d - 1, \text{ and } e := 1 \text{ otherwise}.
\]

**Theorem 26.2.** If \(48 \leq n - 1 \leq m \leq \binom{n}{2} - \left\lceil \frac{n}{2} \right\rceil\), \(k \geq \frac{n}{2}\) and \(C \subseteq \mathcal{P} \subseteq \mathcal{D}_{1+}\), then \(w(n, m, \mathcal{P}) = 2n - k - r - c\).

**Theorem 26.3.** If \(n \geq 15\), \(\binom{n}{2} - \left\lceil \frac{n}{2} \right\rceil + 1 \leq m \leq \binom{n}{2}\) and \(C \subseteq \mathcal{P} \subseteq \mathcal{D}_{1+}\), then \(w(n, m, \mathcal{P}) = 2d + e\).

**Problem 26.1.** Complete the results of Theorem 27.2 for \(n \leq 48\) and those of Theorem 26.3 for \(n \leq 14\).

**Problem 26.2.** Find \(w(n, m, \mathcal{P})\) for other graph properties \(\mathcal{P}\).

**References:**

1. S. Jendrol', I. Schiermeyer, On a Max-Min problem concerning weights of edges
   Combinatorica 21 (2001), 351-359.

27 TURAN TYPE NUMBERS FOR EDGES IN HYPERCUBE GRAPHS

(Heiko Harborth, Hauke Nienborg)

Let \(f(n, k, s)\) denote the maximum number of edges of \(Q_n\) such that at least \(s\) edges of every subgraph \(Q_k\) are missing. Thus \(f(n, k, 1)\) is the Turan type problem of \(P\). Erdős for subgraphs \(Q_k\) of the host graph \(Q_n\). – Especially, determine \(f(n, k, k^{2(k-1)} - 1) = E(n, k)\) being the maximum number of edges of \(Q_n\) such that every subgraph \(Q_k\) contains at most one edge.

**References:**

1. N. Alon, A. Krech, T. Szabo, Turan's theorem in the hypercube

2. H. Harborth, H. Nienborg, Turan numbers relative to platonic solid graphs

28 Minimum number of crossings of pseudo diagonals in convex polygons

(Heiko Harborth)

Consider convex \( n \)-gons with all diagonals as pseudoline segments. Two diagonals with a common vertex do not intersect and two disjoint diagonals intersect at most once. Determine the minimum number \( c(n) \) of crossings if multiple crossings are allowed. So, \( c(7) = 29, c(8) = 49, c(9) = 86, c(10) = 126, c(11) \leq 198, c(12) \leq 273, c(14) \leq 526, \) and \( c(16) \leq 834 \) instead of \( 1820 = \binom{16}{4} \).

29 Multiple crossings in drawings of complete graphs

(Heiko Harborth)

Let \( M(2m) \) denote the maximum number of \( m \)-fold crossings of edges in drawings of the complete graph \( K_{2m} \) (at most one crossing for disjoint edges and no crossings for edges with a common vertex). Prove that \( M(2m) \leq 2 \) for \( m \geq 5 \).

References:

[1] H. Harborth, Drawings of graphs and multiple crossings

30 Crossing regular cycle drawings

(Heiko Harborth)

Crossing \( r \)-regular drawings \( D(G) \) are realizations of a graph \( G \) in the plane where the curves for its edges have at most one point in common, either a crossing or an endpoint and where every edge has exactly \( r \) crossings.

Do crossing \( r \)-regular drawings \( D(C_n) \) of the cycle graph \( C_n \) exist for \( n \leq 5 \) if \( n \) is even for \( 0 \leq r \leq n - 3 \), and if \( n \) is odd for \( 0 \leq r \leq n - 3 \), \( r \) odd? For \( r \leq 9 \) constructions are known in all cases.
Properties of Graphs

References:

31 Ramsey numbers for graph drawings

(Heiko Harborth)

The drawing Ramsey number \( D_r(G) \) is the smallest \( r \) such that every drawing of the complete graph \( K_r \) in the plane (two edges have at most one point in common, either a crossing or a vertex) contains at least one subdrawing of \( G \) having its maximum number of crossings \( CR(G) \).

For \( G = K_s \) the numbers \( D_r(K_3) = 3 \) and \( D_r(K_4) = 5 \) are trivial. Determine \( D_r(K_5) \). It is known \( 10 \leq D_r(K_5) \leq 113 \).

References:

32 Sphere graphs or Kuratowski in \( \mathbb{R}^3 \)

(Heiko Harborth)

In 1930 K. Kuratowski proved the graph planarity criterion: A graph is planar if and only if it does not contain a subgraph homeomorphic to either \( K_5 \) or \( K_{3,3} \).

Planar graphs also are known to be equivalent to coin graphs, that is, to nonoverlapping circles around the vertexpoints such that two circles have a touching point if and only if the corresponding vertices are adjacent.

What about spatial graphs, that is, there are nonoverlapping spheres around the vertexpoints such that two spheres have a touching point if and only if the corresponding vertices are adjacent.

Find a criterion for spatial graphs corresponding to Kuratowski’s criterion for planar graphs. Are \( K_6 \) and \( K_{4,4} \) the graphs corresponding to the Kuratowski graphs \( K_5 \) and \( K_{3,3} \)?
33 STRAIGHT AHEAD CYCLES IN DRAWINGS OF EULERIAN
GRAPHS

(Heiko Harborth)

A path in a drawing $D(G)$ of a graph $G$ (two edges have at most one point in
common, either a crossing or a vertex) passes a vertex straight ahead if it leaves
the same number of edges on both sides.

Problem 33.1. For any Eulerian graph $G$ does there exist a $D(G)$ inducig an Eulerian
straight ahead cycle? - Examples are known for up to 8 vertices, for complete graphs
$K_{2r+1}$, and for complete bipartite graphs $K_{2r,2s}$. What about cube graphs $Q_{2d}$ for
d $\geq 3$?

Problem 33.2. For $G = K_{2r+1}$ with $2r + 1$ congr. 1 or 3$(mod 6)$ does there exist
a $D(K_{2r+1})$ inducing straight ahead triangles only? - An example is known for $K_7$.
Find a $D(K_9)$. 

Paths

34 Minimum $k$-path vertex cover

(Jan Katrenič, Ingo Schiermeyer, Gabriel Semanišin)

Let $G$ be a graph and $k \geq 2$ be an arbitrary, fixed integer. Then $S \subset V(G)$ is called a $k$-path vertex cover of $G$ if every path on $k$ vertices in $G$ contains a vertex from $S$. The cardinality of a minimum $k$-path vertex cover of $G$ is denoted by $\psi_k(G)$. Note that $\psi_2(G)$ is the cardinality of a minimum vertex cover of $G$. The concept of $k$-path vertex cover is related to the algorithm of M. Novotný for a secure communication in wireless sensor networks (see [4]). The invariant $\psi_k(G)$ was introduced in [1] and some exact values and estimations of it are presented in [1], [2]. For example, for $d$-regular graphs we have the following result:

**Theorem 34.1.** Let $k \geq 2$ and $d \geq 1$ be positive integers. Then for any $d$-regular graph $G$ the following holds:

$$\psi_k(G) \geq \frac{d - k + 2}{2d - k + 2}|V(G)|.$$

An associated $k$-Path Vertex Cover Problem (abbreviated by $k$-PVCP) can be formulated in the following way:

**Problem 34.1.** Given a graph $G$ and a positive integer $k$. Find a minimum $k$-path vertex cover of $G$.

Some algorithmic aspects of the $k$-PVCP were studied in [1]. It was proved there that the $k$-PVCP is NP-hard for any fixed integer $k \geq 2$. On the other hand, the value of $\psi_k(G)$ for trees and $\psi_3(G)$ for outerplanar graphs can be computed in linear time. One can easily see, that for an arbitrary positive integer $k$, there is a trivial $k$-approximation of $k$-PVCP. The following non-deterministic improvement is obtained in [3]:

**Theorem 34.2.** For the 3-PVCP there is a deterministic algorithm $DPVC$ of approximation ratio $\max(2, \frac{5}{2} \cdot \left\lceil \frac{\Delta - 1}{2} \right\rceil)$, which runs in time $O(2^\Delta n^{O(1)})$ on a graph $G$ with $n$ vertices and maximum degree $\Delta$. 
A deterministic approach was presented in [6] and [7]:

**Theorem 34.3.** There is a deterministic 2-approximation algorithm for 3-PVCP.

Some algorithms for a modification of the original problem were studied in [5]. The presented results lead us to the following problem:

**Problem 34.2.** Let $k \geq 4$. Is there a $k - 1$ factor approximation algorithm for the $k$-PVCP?

**References:**


Notes