Strong and weak error estimates for the solutions of elliptic partial differential equations with random coefficients

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• $D$ a bounded $C^2$ domain of $\mathbb{R}^d$, $(\Omega, \mathcal{F}, P)$ a probability space

• $a : \Omega \times D \to \mathbb{R}$ a lognormal homogeneous random field
  \[ a(\omega, x) = e^{g(\omega, x)} \] where $g$ is a gaussian homogeneous mean-free random field with $\text{cov}[g](x, y) = k(\|x - y\|), \ k \in C^{0,1}(\mathbb{R})$

• We look for $u : \Omega \times D \to \mathbb{R}$ such that for almost every $\omega$

\[
- \nabla.(a(\omega, .)\nabla u(\omega, .)) = f(x) \text{ on } D \tag{1} \\
\]  
  \[ u(\omega, .) = 0 \text{ on } \partial D. \]
Proposition

For almost all $\omega$, $a(\omega, .) \in C^{0,\alpha}$ for any $\alpha < \frac{1}{2}$. 
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We define for almost all $\omega$:

$$a_{\text{min}}(\omega) = \min_{x \in D} a(\omega, x) \text{ and } a_{\text{max}}(\omega) = \max_{x \in D} a(\omega, x).$$

Then $\frac{1}{a_{\text{min}}(\omega)} \in L^p(\Omega)$ and $a_{\text{max}}(\omega) \in L^p(\Omega)$ $\forall p > 0$. 
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Proposition

The equation 1 admits a unique solution $u \in L^p(\Omega, H^1_0(D))$, $\forall p > 0$. 
Approximation of \( a \)

→ **Approximate \( a \) using a finite number of random variables** is the first step of several numerical methods: stochastic galerkin methods, stochastic collocation method...
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→ We denote by $g_N$ the truncated Karhunen-Loève expansion of $g$ at order $N$:

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g_N(\omega, x) = \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)
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The $(\lambda_n, b_n)$ are the eigenpairs of the Hilbert-Schmidt operator:

$$f \in L^2(D) \quad \mapsto \quad \left( x \mapsto \int_D \text{cov}[g](x, y)f(y)dy \right) \in L^2(D)$$
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**Remark:** Here the $(Y_n)_{n \geq 1}$ are independent because $g$ is gaussian.
The following convergence results are well-known:

\[ g_N \xrightarrow{L^2(\Omega \times D)} g \text{ and, by Mercer theorem } \sup_{x \in D} \|g_N - g\|_{L^2(\Omega)} \xrightarrow{N \to +\infty} 0. \]
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→ We define the approximation \( u_N \) of \( u \) as the solution of:
\[ -\nabla.(a_N(\omega, .) \nabla u_N(\omega, .)) = f(x) \text{ on } D \]
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→ Our aim is to estimate the error commited by approximating \( u \) by \( u_N \).
Strong convergence of $a_N$ to $a$

Assumptions:

- the eigenfunctions $b_n$ are continuously differentiable with $\|b_n\|_\infty \leq C$ and $\|b'_n\|_\infty \leq C n^a$
- $\sum_{n \geq 1} \lambda_n n^b < +\infty$ for some $b > 0$. 
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Strong convergence of $g_N$ to $g$

- $\forall p > 0$, $\forall 0 < \alpha < \min\{b, 2a\}$

$$\|g_N - g\|_{L^p(\Omega, C^0(D))} \leq A_{\alpha, p} \sqrt{\sum_{n > N} \lambda_n n^\alpha} \quad \forall N \in \mathbb{N}.$$
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- For almost all $\omega$, $g_N \overset{C^0(D)}{\longrightarrow} g$ and so $a_N \overset{C^0(D)}{\longrightarrow} a$ as $N \to +\infty$.
- We define $a_N^{\min}(\omega) = \min_{x \in D} a_N(\omega, x)$ and $a_N^{\max}(\omega) = \max_{x \in D} a_N(\omega, x)$ a.s.

Then for all $p > 0$,

\[
\left\| \frac{1}{a_N^{\min}} \right\|_{L^p(\Omega)} \leq B_p \quad \text{and} \quad \|a_N^{\max}\|_{L^p(\Omega)} \leq B_p \quad \forall N \in \mathbb{N}.
\]
Strong convergence of $u_N$ to $u$

$\forall p > 0$, $\forall 0 < \alpha < \min \{ b, 2a \}$

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Weak convergence of $u_N$ to $u$

**Proposition**

There exists a constant $C$ such that for any $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant $C_{\varphi}$

$$\left\| \mathbb{E} \omega [\varphi(u) - \varphi(u_N)] \right\|_{H_0^1} \leq CC_{\varphi}\sum_{n \geq N} \lambda_n.$$
Weak convergence of $u_N$ to $u$

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There exists a constant $C$ such that for any $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant $C_\varphi$

\[ ||E_\omega[\varphi(u) - \varphi(u_N)]||_{H_0^1} \leq CC_\varphi \sum_{n> N} \lambda_n. \]

**Remark:** The weak order is twice the strong order.
Weak convergence of $u_N$ to $u$

**Proposition**

There exists a constant $C$ such that for any $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant $C_\varphi$

$$\|E_\omega[\varphi(u) - \varphi(u_N)]\|_{H^1_0} \leq CC_\varphi \sum_{n> N} \lambda_n.$$ 

**Remark:** The weak order is twice the strong order.

**Sketch of the proof:** We recall that: $u_N(\omega, x) = u_N(Y_1(\omega), ..., Y_N(\omega), x)$ (Doob-Dynkin lemma). For any multi-index $\alpha \in \mathbb{N}^N$ with finite support

$$\left\| \frac{\partial^\alpha u_N(y, x)}{\partial y^\alpha} \right\|_{H^1_0(D)} \leq k_{|\alpha|} \sqrt{\frac{a_N^{\text{max}}(y)}{a_N^{\text{min}}(y)}} \| u_N \|_{H^1_0} C^{|\alpha|} \prod_{i \in \mathbb{N}} \sqrt{\lambda_i^\alpha}.$$
Formally, in the particular case of the expected value, we have:

\[ u(\omega, x) - u_N(\omega, x) = u(Y_1(\omega), \ldots, Y_N(\omega), Y_{N+1}(\omega), \ldots, x) - u(Y_1(\omega), \ldots, Y_N(\omega), 0, \ldots, x) \]
Formally, in the particular case of the expected value, we have:

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\begin{align*}
  u(\omega, x) &- u_N(\omega, x) \\
  &= u(Y_1(\omega), \ldots, Y_N(\omega), Y_{N+1}(\omega), \ldots, x) - u(Y_1(\omega), \ldots, Y_N(\omega), 0, \ldots, x) \\
  &= \sum_{i>N} \frac{\partial u}{\partial y_i}(Y_1(\omega), \ldots, Y_N(\omega), 0, \ldots, x) Y_i(\omega)
\end{align*}
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Formally, in the particular case of the expected value, we have:

\[ u(\omega, x) - u_N(\omega, x) \]
\[ = u(Y_1(\omega), ..., Y_N(\omega), Y_{N+1}(\omega), ..., x) - u(Y_1(\omega), ..., Y_N(\omega), 0, ..., x) \]
\[ = \sum_{i>N} \frac{\partial u}{\partial y_i}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x)Y_i(\omega) \]
\[ + \frac{1}{2} \sum_{i\neq j>N} \frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x)Y_i(\omega)Y_j(\omega) \]
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\[ + \frac{1}{2} \sum_{i > N} \frac{\partial^2 u}{\partial y_i^2} (Y_1(\omega), \ldots, Y_N(\omega), 0, \ldots, x) Y_i(\omega)^2 + \ldots \]
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 & = \sum_{i > N} \frac{\partial u}{\partial y_i}(Y_1(\omega), \ldots, Y_N(\omega), 0, \ldots, x) Y_i(\omega) \\
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\end{align*}
\]

The independence of the \( Y_i \) yields:

\[
\mathbb{E}[u - u_N](x) = 0 + \frac{1}{2} \sum_{i > N} \mathbb{E} \left[ \frac{\partial^2 u}{\partial y_i^2}(Y_1(\omega), \ldots, Y_N(\omega), 0, \ldots, x) \right] + \ldots
\]
Example: the 1D exponential covariance case

We take \( D = (0, 1) \) and \( \text{cov}[g](x, y) = \sigma^2 e^{-\frac{|x-y|}{l}} \) where \( l \) is the correlation length. Then we have analytic expressions for the eigenvalues \( \lambda_n \) and the eigenfunctions \( b_n \), in particular:

- \( \lambda_n \xrightarrow{n \to +\infty} \frac{2\sigma^2}{\pi^2 n^2} \)
- \( \forall n \in \mathbb{N}, \|b_n\|_{\infty} \leq C \) and \( \|b'_n\|_{\infty} \leq Cn. \)
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Proposition (Strong convergence result)

\[ \forall p > 0, \forall 0 < \alpha < 1 \]

\[ \|u_N - u\|_{L^p(\Omega, H^1_0(D))} \leq F_{\alpha, p} N^{\frac{\alpha-1}{2}} \quad \forall N \in \mathbb{N}. \]
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Proposition (Weak convergence result)

For any $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant $C_\varphi$

$$\|\mathbb{E}_\omega [\varphi(u) - \varphi(u_N)]\|_{H^1_0(D)} \leq C_\varphi \frac{C}{N}.$$
\( a_N(\omega, x) \) for different values of \( N \)

\[ \mathbb{E}[u_N(x)] \] for different values of \( N \)

here we have \( \| E[u - u_N] \|_{L^2(D)} \approx \frac{c}{N^{2.7}} \).
$\mathbb{E}[u_N(x)]$ for different values of $N$, in the case where $l = 0.1$, $\sigma = 1$. 
Example: the analytic covariance case

We suppose that $\text{cov}[g]$ is analytic on $D^2$, then we have

**Theorem (Schwab, Todor)**

- $\lambda_n \leq c_1 e^{-c_2 n^{1/d}} \quad \forall n \in \mathbb{N}$

- for any $s > 0$ there exists a constant $c_s$ such that,

\[
\|b_n\|_\infty \leq c_s |\lambda_n|^{-s} \quad \text{and} \quad \|b'_n\|_\infty \leq c_s |\lambda_n|^{-s} \quad \forall n \in \mathbb{N}.
\]

We have then strong and weak convergence results, analogous to the previous results.
Proposition (Strong convergence result)

For any $0 < s < \frac{1}{2}$, and $p > 0$

$$\| u - u_N \|_{L^p(\Omega,H^1_0(D))} \leq H_{s,p} \sqrt{\sum_{n>N} \lambda_n^{1-2s}} \quad \forall N \in \mathbb{N}$$

therefore

$$\| u - u_N \|_{L^p(\Omega,H^1_0(D))} \leq l_{d,s,p} N^{\frac{d-1}{2d}} e^{-\frac{c_2(1-2s)}{2}} N^{1/d} \quad \forall N \in \mathbb{N}$$
Proposition (Weak convergence result)

For any $0 < s < \frac{1}{2}$, for all $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant $C_\varphi$, we have:

$$\|E[\varphi(u_N) - \varphi(u)]\|_{H^1_0(D)} \leq J_s C_\varphi \sum_{n>N} \chi_n^{1-2s} \quad \forall N \in \mathbb{N}$$

therefore

$$\|E[\varphi(u_N) - \varphi(u)]\|_{H^1_0(D)} \leq K_{d,s} C_\varphi N^{\frac{d-1}{d}} e^{-c_2(1-2s)N^{1/d}} \quad \forall N \in \mathbb{N}. $$