Fast digital curvature estimation

*Mathematical Optics, Image Modelling and Algorithms*

20. Juni 2016
Curvature describes how far a curve locally deviates from its tangent.
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The curvature is defined by

$$\kappa(t) = \frac{\det(\dot{\alpha}(t), \ddot{\alpha}(t))}{\|\dot{\alpha}(t)\|^3}.$$
Outline of the talk

- Curvature Estimation
- Digital Curvature Estimation
  - Variation-diminishing Approximation
  - Variation-diminishing Splines
  - Multiscale Algorithm
- Numerical Evaluation
- Summary
Curvature Estimation

Sufficient conditions for uniform convergence
Curvature Estimation

The curvature of a planar $C^2$-curve

Let $\alpha(t) = (x(t), y(t))^T$.

The signed curvature is defined by

$$\kappa(t) = \frac{\text{det}(\dot{\alpha}(t), \ddot{\alpha}(t))}{\|\dot{\alpha}(t)\|^3}.$$
Let $\alpha(t) = (x(t), y(t))^T$.

The signed curvature is defined by

$$\kappa(t) = \frac{\det(\dot{\alpha}(t), \ddot{\alpha}(t))}{\|\dot{\alpha}(t)\|^3}.$$ 

We approximate the signed curvature by

$$\kappa_n(t) = \frac{\det(\dot{\tilde{\alpha}}_n(t), \ddot{\tilde{\alpha}}_n(t))}{\|\dot{\tilde{\alpha}}_n(t)\|^3},$$

where $\tilde{\alpha}_n : [0, 1] \rightarrow \mathbb{R}^2$ is a smooth $n$-term approximation of $\alpha$. 
Theorem [N., 2015]

Let \( \alpha : I \to \mathbb{R}^2 \) be a regular planar \( C^2 \)-curve with smooth approximations \( \tilde{\alpha}_n \). Suppose that

\[
\lim_{n \to \infty} \| \alpha(t) - \tilde{\alpha}_n(t) \|_\infty = 0, \quad \lim_{n \to \infty} \| \dot{\alpha}(t) - \dot{\tilde{\alpha}}_n(t) \|_\infty = 0
\]

and there exist constants \( C_3, C_4 > 0 \) independent on \( n \) such that

\[
\| \dot{\tilde{\alpha}}_n \|_\infty \leq C_3 \| \dot{\alpha} \|_\infty, \quad \| \ddot{\tilde{\alpha}}_n \|_\infty \leq C_4 \| \ddot{\alpha} \|_\infty.
\]

Then there exist \( C_1, C_2 > 0 \), such that

\[
|\kappa(t) - \kappa_n(t)| \leq C_1 \cdot \left( |\dot{x}(t) - \dot{x}_n(t)| + |\dot{y}(t) - \dot{y}_n(t)| \right)
+ C_2 \cdot \left( |\ddot{x}(t) - \ddot{x}_n(t)| + |\ddot{y}(t) - \ddot{y}_n(t)| \right),
\]

for \( n \to \infty \) and all \( t \in [0, 1] \).

The constants \( C_1, C_2 \) depend only on \( n, \| \alpha \|_\infty, \| \dot{\alpha} \|_\infty \) and \( \| \ddot{\alpha} \|_\infty \).
Corollary [N., 2015]

Let \((\tilde{\alpha}_n)_{n\in\mathbb{N}}\) be a sequence of \(C^2\)-smooth approximations of \(\alpha \in C^2([0, 1], \mathbb{R}^2)\) such that for \(n \to \infty\)

- \(\tilde{\alpha}_n \to \alpha\) uniformly,
- \(\tilde{\alpha}_n \to \dot{\alpha}\) uniformly,
- \(\tilde{\alpha}_n \to \ddot{\alpha}\) uniformly,

and there exist constants \(C_1, C_2 > 0\) independent of \(n\) such that

\[
\|\tilde{\alpha}_n\|_\infty \leq C_3 \|\dot{\alpha}\|_\infty \quad \text{and} \quad \|\ddot{\alpha}_n\|_\infty \leq C_4 \|\ddot{\alpha}\|_\infty.
\]

Then for \(n \to \infty\) the approximation of the curvature converges uniformly, i.e.,

\[
\|\kappa(t) - \tilde{\kappa}_n(t)\|_\infty \to 0.
\]
Digital Curvature Estimation

Variation-diminishing approximation with splines
<table>
<thead>
<tr>
<th>Year</th>
<th>Author(s)</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1975</td>
<td>Bennet, Mac Donald</td>
<td>Angular change of the tangent</td>
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<tr>
<td>1986</td>
<td>Asada, Brady</td>
<td>Curvature primal sketch</td>
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<td>1987</td>
<td>Medioni, Yasumoto</td>
<td>Cubic B-splines</td>
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<tr>
<td>1992</td>
<td>Mokhtarian, Mackworth</td>
<td>Multiscale shape representation</td>
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<tr>
<td>1993</td>
<td>Worring, Smeulders</td>
<td>Digital curvature estimation</td>
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<tr>
<td>2001</td>
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<tr>
<td>2003</td>
<td>Utcke</td>
<td>Numerical Error bounds</td>
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<tr>
<td>2007</td>
<td>Hermann, Klette</td>
<td>Comparative study</td>
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**BC Estimator**

<table>
<thead>
<tr>
<th>Year</th>
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<tbody>
<tr>
<td>2008</td>
<td>Malgouyres, Brunet and Fourey</td>
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<td>2011</td>
<td>Esbelin, Malgouyres and Cartade</td>
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**MDCA Estimator**

<table>
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</tr>
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<tr>
<td>2007</td>
<td>De Vieilleville, Lachaud and Feschet</td>
</tr>
<tr>
<td>2011</td>
<td>Roussillon, Lachaud</td>
</tr>
<tr>
<td>2014</td>
<td>Levallois, Coeurjolly and Lachaud</td>
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We want to

- estimate the curvature
- detect critical points

piecewise $C^2$-curve
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piecewise $C^2$-curve
digitized curve
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$C^2$-Interpolation?
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piecewise $C^2$-curve
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$C^2$-Interpolation

$C^2$-Approximation
We want an approximation that mimics the shape of our data, i.e.,

- preserves monotonicity,
- preserves convexity.

To estimate the curvature we need

- a $C^2$-smooth approximation,
- a numerically stable and efficient computation of the 1st and 2nd derivative,
- a fast algorithm.
Let \( \{\varphi_1, \ldots, \varphi_n\} \) be a basis of continuous functions over \([0, 1]\), and let \(0 = \xi_1 < \cdots < \xi_n = 1\) be a partition of \([0, 1]\).

I.J. Schoenberg. Über variationsvermindernde lineare Transformationen. 1930.
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For a continuous function \( f \) on \([0, 1]\), the approximation

\[
T_n(f; t) := \sum_{j=1}^{n} f(\xi_j) \varphi_j(t)
\]

is said to be variation-diminishing if

\[
v(T_n f) \leq v(f),
\]

where \( v \) counts the number of strict sign changes in \([0, 1]\).

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I.J. Schoenberg. Über variationsvermindernde lineare Transformationen. 1930.
If the basis is normalized, i.e.,

$$\sum_{j=1}^{n} \varphi_j(t) = 1, \quad \text{for all } t \in [0, 1],$$

and there exists $0 = \xi_1 < \cdots < \xi_n = 1$ such that

$$\sum_{j=1}^{n} \xi_j \varphi_j(t) = t, \quad \text{for all } t \in [0, 1],$$

then we obtain due to the variation-diminishing property:

- $T_n$ preserves the \textit{positivity}
- $T_n$ preserves the \textit{monotonicity}
- $T_n$ preserves the \textit{convexity}

Let $n, k > 0$ be integers and $\Delta_n = \{t_j\}_{j=-k}^{n+k}$ a partition of $[0, 1]$ such that

$$0 = t_{-k} = \cdots = t_0 < t_1 < \cdots < t_n = \cdots = t_{n+k} = 1.$$ 

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For $f \in C([0, 1])$, the *Schoenberg operator* of degree $k$ is defined by

$$S_{\Delta_n, k} f(t) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(t), \quad 0 \leq t \leq 1$$

with Greville nodes

$$\xi_{j,k} := \frac{t_{j+1} + \cdots + t_{j+k}}{k}, \quad -k \leq j \leq n-1,$$

and normalized B-splines $N_{j,k}(t)$.

Let \( f \in C^2([0, 1]) \) and \( k > 2 \). Then the spline approximation \( \kappa_{\Delta_n,k} \) is well-defined:

**Lemma [Marsden, 1970]**

- \( \lim_{n \to \infty} S_{\Delta_n,k}(f; t) = f(t) \) uniformly on \([0, 1]\)
- \( \lim_{n \to \infty} D S_{\Delta_n,k}(f; t) = Df(t) \) uniformly on \([0, 1]\)
- \( \lim_{n \to \infty} D^2 S_{\Delta_n,k}(f; t) = D^2 f(t) \) uniformly on \((0, 1)\)

**Lemma [N., 2015]**

The estimates of the derivatives of the spline approximations are bounded by

- \( \|D S_{\Delta_n,k} f\|_{\infty} \leq \|D f\|_{\infty} \),
- \( \|D^2 S_{\Delta_n,k} f\|_{\infty} \leq (k - 1) \frac{\delta_{\max}}{\delta_{\min}} \cdot \|D^2 f\|_{\infty} \).
We consider the following multiscale algorithm at scales $s_1 < \cdots < s_l$:

1. Compute the cubic spline approximation $\tilde{\alpha}, \tilde{\alpha}^{(s_1)}, \ldots, \tilde{\alpha}^{(s_l)}$
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1. Compute the cubic spline approximation $\tilde{\alpha}, \tilde{\alpha}(s_1), \ldots, \tilde{\alpha}(s_l)$
2. Compute the curvature estimates $\tilde{\kappa}_{n}(s_j)(\alpha; t)$
3. Measure the absolute approximation error $\varepsilon_j(t_i) := |\tilde{\alpha}^{s_j}(t_i) - \alpha(t_i)|$
Digital Curvature Estimation

A multiscale algorithm

We consider the following multiscale algorithm at scales \( s_1 < \cdots < s_l \):

1. Compute the cubic spline approximation \( \tilde{\alpha}, \tilde{\alpha}^{(s_1)}, \ldots, \tilde{\alpha}^{(s_l)} \)
2. Compute the curvature estimates \( \tilde{\kappa}_n^{(s_j)}(\alpha; t) \)
3. Measure the absolute approximation error \( \varepsilon_j(t_i) := |\tilde{\alpha}^{s_j}(t_i) - \alpha(t_i)| \)
4. Compute the discrete decay rates by

\[
d^j_{j-1}(t_i) := |\varepsilon_{s_{j-1}}(t_i) - \varepsilon_{s_j}(t_i)|, \quad j \in \{2, \ldots, l\}
\]
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1. Compute the cubic spline approximation $\tilde{\alpha}, \tilde{\alpha}^{(s_1)}, \ldots, \tilde{\alpha}^{(s_l)}$

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$$d^j_{j-1}(t_i) := |\epsilon_{s_{j-1}}(t_i) - \epsilon_{s_j}(t_i)|, \quad j \in \{2, \ldots, l\}$$

5. The final curvature estimate is given by

$$\tilde{\kappa}_n(\alpha; t_i) := \sum_{j=1}^{l-1} \frac{\left( \epsilon_j(t_i) \cdot d^{j+1}_j(t_i) \right)^{-1} \kappa_n^{(s_j)}(t_i)}{\sum_{j=1}^{l-1} \left( \epsilon_j(t_i) \cdot d^{j+1}_j(t_i) \right)^{-1}}.$$
Numerical Evaluation

Comparison with the DC and the MDCA curvature estimator
Digitized ellipse with great axis 30 and small axis 20 rotated by 0.5 rad.

(a) $h = 1$ (104 pixels)  
(b) $h = 0.1$ (1016 pixels)  
(c) curvature profile
Numerical Evaluation

Flower

Digitized flower with 5 petals with great radius 20 and small radius 5.

(a) $h = 1$ (368 pixels)

(b) $h = 0.1$ (3644 pixels)

(c) curvature profile
Numerical Evaluation

Pointwise curve evaluation

Curvature error of an ellipse

\[ C \cdot n^{-2} \]
Numerical Evaluation

Pointwise curve evaluation

Curvature error of a flower

max. error
avg. error
$C \cdot n^{-2}$

absolute error

number of samples $n$

Curvature error of a flower

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Numerical Evaluation

Working with real data

“In theory there is no difference between theory and practice. In practice there is.”

Yogi Berra
Numerical Evaluation

Working with real data

“In theory there is no difference between theory and practice. In practice there is.”

Yogi Berra

Problems

- digitization instead of correct point evaluations
- the variation-diminishing property is not “working”
Numerical Evaluation

Working with real data

“In theory there is no difference between theory and practice. In practice there is.”
Yogi Berra

Problems

- digitization instead of correct point evaluations
- the variation-diminishing property is not “working”

Solution: Smoothing
Numerical Evaluation

Comparison of relative errors: Ellipse

(a) Relative average error

(b) Relative maximal error

The relative average and maximal errors.
Numerical Evaluation

Comparison of relative errors: Flower

The relative average and maximal error.
<table>
<thead>
<tr>
<th></th>
<th>relative avg. error</th>
<th>run time [ms]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BC</td>
<td>MDCA</td>
</tr>
<tr>
<td>Ellipse, $h = 1$</td>
<td>0.0930</td>
<td>0.1138</td>
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<tr>
<td>Ellipse, $h = 0.1$</td>
<td>0.0298</td>
<td>0.0311</td>
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<tr>
<td>Ellipse, $h = 0.01$</td>
<td>0.0105</td>
<td><strong>0.0090</strong></td>
</tr>
<tr>
<td>Flower, $h = 1$</td>
<td>0.3875</td>
<td><strong>0.3354</strong></td>
</tr>
<tr>
<td>Flower, $h = 0.1$</td>
<td>0.2240</td>
<td><strong>0.0907</strong></td>
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<tr>
<td>Flower, $h = 0.01$</td>
<td>0.0968</td>
<td>0.0266</td>
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</tbody>
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Comparison between curvature estimators.
Numerical Evaluation
Detection of singularities

Rectangle

Singularities

- smooth
- singularity

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Summary

The spline based curvature approximation
- converges numerically stable,
- has competitive accuracy,
- is fast to compute, and
- can deal with piecewise $C^2$-curves.
Summary

The spline based curvature approximation converges numerically stable, has competitive accuracy, is fast to compute, and can deal with piecewise $C^2$-curves.

Open Questions

- lower estimates for $n$-term approximation error
- approximation with convolution operators
- extension to surfaces