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Construction of wavelets on compact manifolds based on conformal mappings

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Introduction

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Wavelet analysis on $\mathbb{R}^2$

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Some historical facts

Continuous wavelet transform of frequency breakdown signal

- first beginnings in Alfréd Haar’s thesis about orthogonality (1910)
- ca. 1946: Dennis Gábor introduced Gabor atoms $\rightsquigarrow$ Gabor analysis
- ca. 1980: modern version, developed by Jean Morlet, Alex Grossmann, George Zweig (in a second phase by Ingrid Daubechies and Stéphan Mallat)

\[^1\text{see https://en.wikipedia.org/wiki/Continuous_wavelet_transform, retrieved on 19 Apr 2016}\]
What are our aims?

Key question: How one can use the power of wavelet analysis on general manifolds?

1. a brief revision of the usual wavelet theory in $\mathbb{R}^n$ in connection with the group theoretical background in the most cases not transferable on manifolds and curved surfaces!

2. in search of solutions: the projective approach to wavelets on manifolds "lift" the wavelet theory from $\mathbb{R}^2$ to the manifold or in other words: we build wavelets and a wavelet transformation with the aid of suitable conformal projections

3. examples with Mathematica v10.3
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Conformal maps and their properties

Definition

A map $f : (M, g) \to (\tilde{M}, \tilde{g})$ between two Riemannian manifolds is called **conformal** if

$$f^* \tilde{g} = \tilde{g}(df, df) = \sigma^2 g$$

where $\sigma \in C^\infty(M), \quad \sigma > 0$

The expression $f^* \tilde{g} \equiv \tilde{g}_{f(p)}(df_p(v), df_p(w))$ is called the induced metric on $M$.

With the aid of $g$ one may define angles and further follows:

Lemma

*Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be a map between Riemannian manifolds. Then $f$ is conformal if and only if $f$ is angle-preserving.*
### Definition

A $n$-dimensional Riemannian manifold $(M, g)$ is **conformally flat** if it can be covered by neighborhoods $\{U_\alpha\}$ such that there exists a conformal map $\phi_\alpha : U_\alpha \to V \subset \mathbb{R}^n$ for every neighborhood $U_\alpha$.

An important result in two dimensions was found in 1916:

### Theorem (Korn and Lichtenstein)

*Any two-dimensional Riemannian manifold is conformally flat.*
Conformal maps on rotationally symmetric surfaces

For this we suppose a surface $S$ which is formed by rotating a curve
$\{(x, z) = (u(\theta), v(\theta)), 0 \leq \theta \leq \pi\}$ in the $x$-$z$-plane around the $z$ axis:

$S = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z)^T = X(\phi, \theta), \quad \phi \in [0, 2\pi], \quad \theta \in [0, \pi]\}$

The parametrization is given by

$X(\phi, \theta) = \begin{pmatrix}
\cos(\phi)u(\theta) \\
\sin(\phi)u(\theta) \\
v(\theta)
\end{pmatrix}$

Next, we take previous definitions into account and consider the
Riemannian metric $g_p$.

**Note:**

- The **first fundamental form** $g_p(w, w) = \|w\|^2_2$ is sufficient.
- We can describe an element $w \in T_pS$ of the tangent space
in terms of a curve $\gamma : (-\epsilon, \epsilon) \rightarrow S$ satisfying $\gamma(0) = p$ and
$\gamma'(0) = w$
By $f : \mathbb{R}^3 \ni S \to \mathbb{R}^2$ we want to denote the conformal map between $S$ and the plane.

We require additionally that $f \circ X$ satisfies the following mapping rule

$$ (f \circ X)(\phi, \theta) = r(\theta) \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad r > 0 $$

**Geometrical interpretation**

All points on the great circle $S \cap \{\phi = \text{const}\}$ are mapped onto a half-line $\{t(\cos \phi, \sin \phi)^T, t \geq 0\}$ in $\mathbb{R}^2$ with respect to a rescaling function $r = r(\theta) > 0$.

**Example:** sphere $S^2$ with the stereographic projection

A straightforward calculation yields an ordinary differential equation for $r$ and the solution reads

$$ r(\theta) = C \exp \left( \int^\theta \frac{\sqrt{u'(\nu)^2 + v'(\nu)^2}}{u(\nu)} d\nu \right), \quad C \in \mathbb{R}^+ $$
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Idea:

- Elements of a group can be represented by transformations of other mathematical objects.
- Starting point: algebraic groups that are equipped with a topology.

Definition

Let $G = (G, \circ)$ be a topological group, $H$ a linear space, $\pi : G \to \mathcal{L}(H)$ a homomorphism (structure preserving) mapping into the group $\mathcal{L}(H)$ of continuous linear operators from $H$ into itself, i.e. we require

$$\pi(g_1 \circ g_2) = \pi(g_1)\pi(g_2) \quad \text{and} \quad \pi(e) = id_H$$

Then the pair $(H, \pi)$ is called group representation of $G$ in $H$. 
Definition

Let $\pi$ be a continuous unitary representation of the group $G$ in a Hilbert-space $H$. An element $v \in H$ is called an **admissible vector**, if

$$\int_G |\langle v, \pi(g)v \rangle_H|^2 d\mu(g) < \infty$$

In the case that there exists an admissible nontrivial element $v$ and $\pi$ is irreducible, we call $\pi$ **square-integrable**.
Definition

Let $\mathcal{G} = (G, \circ)$ be a locally compact, topological group and $\pi$ a square-integrable unitary representation of $\mathcal{G}$ in a Hilbert-space $H$. Further let $f \in H$. An element $\psi \in H \setminus \{0\}$ is called wavelet if $\psi$ is an admissible vector. The (left) wavelet transform of $f$ is defined by

$$L_\psi : H \to L^2(G), \quad L_\psi(f)(g) = \langle f, \pi(g)\psi \rangle_H$$
Construction of the continuous wavelet transform on $\mathbb{R}^2$

Given a function $f \in L^2(\mathbb{R}^2, d^2x)$. We can define motions $T_b : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $R_\alpha : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ and the dilation $D_a : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$T_b f(x) = f(x - b), \quad b \in \mathbb{R}^2$$
$$R_\alpha f(x) = f(r_{-\alpha}x), \quad r_{-\alpha} \in SO(2)$$
$$D_a f(x) = a^{-1}f(a^{-1}x), \quad a \in \mathbb{R}^+ := (0, \infty)$$

where $r_\phi$ takes the form $\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$.

The parameters $b, a$ and $\alpha$ themselves define the (affine) Euclidean group $\mathcal{G} = (\mathbb{R}^2, +) \times (\mathbb{R}^+, \cdot) \times (\mathbb{R}, +) =: (\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}, \ast)$. This group is characterized by the composition of $T_b, D_a$ and $R_\alpha$. Since we have

$$(T_{b_1} D_{a_1} R_{\alpha_1})(T_{b_2} D_{a_2} R_{\alpha_2}) f(x) = T_{b_1 + a_1 r_{-\alpha_1} b_2} D_{a_1 a_2} R_{\alpha_1 + \alpha_2} f(x)$$

the group law in $\mathcal{G}$ reads

$$(b_1, a_1, \alpha_1) \ast (b_1, a_1, \alpha_1) = (b_1 + a_1 r_{-\alpha_1} b_2, a_1 a_2, \alpha_1 + \alpha_2)$$
Let $\psi \in L^2(\mathbb{R}^2)$ be a (complex-valued) wavelet. Then the left wavelet transform $L_{\psi} : L^2(\mathbb{R}^2) \rightarrow L^2(G, d\mu_L)$ is given by

$$L_{\psi}(f)(g) = \langle f, \pi(g)\psi \rangle_{L^2(\mathbb{R}^2)} = a^{-1} \int_{\mathbb{R}^2} \overline{\psi(a^{-1}r_\alpha(x-b))} f(x) d^2x$$

where $d\mu_L = a^{-3} d^2b da d\alpha$ denotes the left invariant Haar-measure on $G$. It exists uniquely up to a constant.
The two-dimensional discrete wavelet transform

The discrete wavelet transformation (DWT) is based on a multi-resolution analysis (MRA) of our function space $L^2(\mathbb{R}^2)$. We introduce the two-dimensional MRA as a natural extension from 1D.

\[ \ldots \leftarrow V_{-2} \leftarrow V_{-1} \leftarrow V_0 \leftarrow V_1 \leftarrow V_1 \leftarrow \ldots \]
\[ \ldots \nearrow W_{-2} \nearrow W_{-1} \nearrow W_0 \nearrow W_1 \nearrow W_1 \nearrow \ldots \]

The sequence of closed subspaces must satisfy

\[ \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \text{ and } \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^2) \]

together with

(i) $f(x) \in V_j \iff f(Dx) \in V_{j+1}$

(ii) There exists a function $\phi \in V_0$ such that $\{\phi(x-k), k \in \mathbb{Z}^2\}$ is an orthonormal basis of $V_0$. 
A particular two-dimensional MRA can be obtained by taking the direct product of two one-dimensional structures:

We pick a MRA of $L^2(\mathbb{R})$, namely $\{V_j, j \in \mathbb{Z}\}$, and get the new MRA

$$\{(2)V_j := V_j \otimes V_j, j \in \mathbb{Z}\}$$

**Consequences:**

- The MRA is associated to the dilation matrix $D = \text{diag}(2, 2)$.
- We need one scaling function $\Phi(x,y) := \phi(x)\phi(y) \in (2)V_0$, but
- obtain three wavelet spaces $(2)W^h_0$, $(2)W^v_0$ and $(2)W^d_0$.

However, for practical purposes (image processing,...) a finite decomposition is necessary (the signal $f$ is taken in some $V_J$):

$$V_J = V_{j_0} \oplus \bigoplus_{j=j_0}^{J-1} W_j, \quad j_0 < J$$

**question:** Which $J$ should I choose? $\rightsquigarrow$ "wavelet crime"
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Manifolds and the group theoretical approach

Let us assume that data are given on smooth and compact two-dimensional manifold $M$.

**Problem:** motions on a general manifold do not have the structure of a group ($M$ does not admit a global isometry group)

From now on we assume that $M$ admits such a global isometry group. We additionally need:

- dilations: are defined via the tangent plane $\mathcal{P}$
  Let $p : M \to \mathcal{P}$ be a bijective projection. Three steps are necessary:
  
  (i) project each point $\xi \in M$ on $\mathcal{P}$
  (ii) perform the usual dilation
  (iii) lift the result back onto $M$ by $p^{-1}$.

- parameter group: built of groups of motions and dilations
  Since the existence of an outer automorphism is not always given, the semidirect product is no longer useful for constructing the parameter group!
The situation will be less complicated if we consider the two-sphere (motions are elements of the rotation group $SO(3)$ and the semidirect product is replaced by a group factorization):

Real part of the spherical Morlet wavelet  
Imaginary part of the spherical Morlet wavelet

another way to work around all mentioned problems: **local wavelet transform**
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Let $M \subset \mathbb{R}^3$ be a $C^1$-surface defined by the parametrization

$$M = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : (\xi_1, \xi_2, \xi_3)^T = X(\phi, \theta), \ \phi \in I_\phi \land \theta \in I_\theta\}$$

where $I_\phi$ and $I_\theta$ are intervals and $X \in C^1(I_\phi \times I_\theta)$. The length of the normal is represented by a function $L : I_\phi \times I_\theta \to \mathbb{R}^+ \cup \{0\}$,

$$L(\phi, \theta) = \|X_\phi \times X_\theta\|_2 = \left\| \frac{\partial X}{\partial \phi} \times \frac{\partial X}{\partial \theta} \right\|_2,$$

and the area element on $M$ is

$$d\mu(\xi) = d\mu(\phi, \theta) = L(\phi, \theta)d\phi\,d\theta.$$
Next, we consider a global bijective projection $p : M \to \mathbb{R}^2$.

$$d\mathbf{x} = \left| \frac{\det J_p(\phi, \theta)}{L(\phi, \theta)} \right| d\mu \quad \text{and} \quad d\mu = \frac{(L \circ p^{-1})(x, y)}{\left| \det(J_p \circ p^{-1})(x, y) \right|} d\mathbf{x}$$

$$=: \nu(\phi, \theta)$$

$$=: \rho(x, y)$$

The functions $\nu : M \to \mathbb{R}^+ \cup \{0\}$ and $\rho : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ are weights and $J_p$ denotes the Jacobian of $p$.

In $L^2(M, d\mu)$ we define the inner product $\langle \cdot, \cdot \rangle_\ast$ as

$$\langle \tilde{f}, \tilde{g} \rangle_\ast := \langle \tilde{f} \circ p^{-1}, \tilde{g} \circ p^{-1} \rangle_{L^2(\mathbb{R}^2)} \quad \text{for all } \tilde{f}, \tilde{g} \in L^2(M, d\mu)$$

This definition yields further

$$\langle f \circ p, g \circ p \rangle_\ast = \langle f \circ p \circ p^{-1}, g \circ p \circ p^{-1} \rangle_{L^2(\mathbb{R}^2)} = \langle f, g \rangle_{L^2(\mathbb{R}^2)}$$
Proposition

Let $\Pi : L^2(M, d\mu) \to L^2(\mathbb{R}^2, d\mathbf{x})$ be an operator defined by

$$\Pi \tilde{f} := \rho \cdot (\tilde{f} \circ p^{-1}), \quad \text{for all } \tilde{f} \in L^2(M, d\mu)$$

Its inverse reads then $\Pi^{-1} f = \nu \cdot (f \circ p), \quad f \in L^2(\mathbb{R}^2, d\mathbf{x})$. Moreover $\Pi$ (or $\Pi^{-1}$, respectively) is an unitary operator.

Consequences:

- orthogonality in $L^2(\mathbb{R}^2)$ is transferred to $L^2(M)$
- formulation of the lifted CWT without big effort:

$$L^M_{\tilde{\psi}}(\tilde{f})(g) := \langle \tilde{f}, (\pi(g)\psi) \circ p \rangle_\ast$$

But $\tilde{\psi} = \psi \circ p$ is strictly speaking no admissible vector in the Hilbert-space $L^2(M)$!

- The family of sets $\{V_j := \{f \circ p, f \in V_j\}, j \in \mathbb{Z}\}$ yields a multi-resolution analysis of $L^2(M)$ with the same dilation matrix.
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Examples

We consider the function $g(\phi, \theta + \theta_0) \in L^2(M)$ defined by

$$
g(\phi, \theta) := \begin{cases} 
1, & \theta \geq \frac{\pi}{2} \\
(1 + 3 \cos^2 \theta)^{-1/2}, & \theta < \frac{\pi}{2}
\end{cases}
$$

where $\theta_0 \in [0, \pi]$ is an arbitrary constant.

The second partial derivative with respect to $\theta$ has a discontinuity on the circle $\theta = \frac{\pi}{2} - \theta_0$.

Let us try to detect this discontinuity!

Remark

Since a software-based implementation requires signals with finite length, we represent our signal by a matrix $F = (f_{i,j})$. It does not matter whether the signal is defined on $\mathbb{R}^2$ or on the manifold $M$ because in both cases we can discretize the signal to the same matrix.
The function $g(\phi, \theta + \theta_0)$...

Data on the sphere with $\theta_0 = 40^\circ$

Data on the rotated nephroid with $\theta_0 = 40^\circ$
...and the resulting DWT

DWT on the sphere, $\theta_0 = 40^\circ$

DWT on the rotated nephroid, $\theta_0 = 40^\circ$
A distorted version of the first example is presented here:

Data on the modified sphere, 
\[ \theta_0 = 35^\circ \]

DWT on the modified sphere, 
\[ \theta_0 = 35^\circ \]

The critical circle is also distinctly recognizable!
Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.

David Hilbert (1862 - 1943)

Thank you for your attention!
Riemannian manifolds

Riemannian manifolds provide the basis for our work:

**Definition**

A **Riemannian manifold** \((M, g)\) is a \(C^\infty\)-manifold \(M\) equipped with a function \(g\) that maps each point \(p \in M\) onto an inner product \(g_p\) on the tangent space \(T_p M\). More precisely there is a positive definite symmetric bilinear form

\[
g_p : T_p M \times T_p M \to \mathbb{R}
\]

such that the function

\[
M \ni p \mapsto g_p(X_p, Y_p) \in \mathbb{R}
\]

varies smoothly from \(p \in M\), where \(X, Y \in \mathfrak{X}(M)\) are differentiable vector fields on \(M\). The so-called **Riemannian metric** \(g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)\) is the family of inner products \(g_p\).
Example (stereographic projection and the sphere)

We consider the unit sphere $S^2$ around the center $(0, 0, 1)^T \in \mathbb{R}^3$ together with the stereographic projection $p$ from the North pole $N = (0, 0, 2)^T$, i.e. each point $P \in S^2$ is mapped onto the image $P' \in \mathbb{R}^2$ by $p$.

With the aid of the Intercept theorem we follow immediately

$$r(\theta) = 2 \exp \left( \int_{\nu}^{\theta} \frac{\sqrt{\cos(\nu)^2 + \sin(\nu)^2}}{\sin(\nu)} d\nu \right) = 2 \tan \left( \frac{\theta}{2} \right)$$

But on the other hand (with $u(\theta) = \sin(\theta)$, $v(\theta) = 1 - \cos(\theta)$, $C = 2$):

$$r(\theta) = \frac{\sin(\theta)}{2 - (1 - \cos(\theta))} = \tan \left( \frac{\theta}{2} \right)$$
Example (mexican-hat wavelet)

It is defined by the Laplacian of a Gaussian, i.e.

\[ \psi_H(x, y) = -\Delta \exp \left( -\frac{1}{2}(x^2 + y^2) \right) \]