Diploma-Thesis

Colourings of graphs with prescribed cycle lengths

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Selected symbols and notations

C	a cycle	9
$\mathscr{C}$	a colouring of a graph	11
$\mathscr{C}_k$	a colouring with k colours	11
C_k	a cycle of length k	9
$C_e(G)$	set of even cycle lengths of a graph G	21
$C_o(G)$	set of odd cycle lengths of a graph G	17
C^*_{2m+1}	a (2m+1)-cycle with at least 2 diagonals	34
$d(x)$	degree of a vertex x	
$d(x,y)$	distance from vertex x to vertex y	9
$\delta(G)$	minimal degree of a graph G	8
$\Delta(G)$	maximal degree of a graph G	8
E, $E(G)$	edge set of a graph G	7
G	a graph	7
G'	subgraph of G induced by $V(G) - V(C^*_{2m+1})$	34
$G[W]$	induced subgraph by W	8
K_n	a complete graph of order n	9
$N(x)$	neighbourhood of a vertex x	8
n, $n(G)$	order of a graph G	7
P	a path	8

P_k	a path of length k	9
$P(x_0, x_k)$	a path from vertex x_0 to vertex x_k	8
V, $V(G)$	vertex set of a graph G	7
$\chi(G)$	chromatic number of a graph G	12
$\omega(G)$	clique number of a graph G	12

Introduction

This thesis deals with graphs G, a structure consisting of vertices and edges joining them, and (vertex) colourings of these graphs.

In 1941 Brooks answered the question in his paper [Bro41], how many colours one needs in order to guarantee a proper colouring of G. But unfortunately his answer was only an upper bound. Furthermore, the problem of finding an optimal colouring, i.e., a colouring with as few as possible colours, turned out to be *NP-hard*.

But nevertheless, for some classes of graphs it was possible to refine Brooks' bound, and even so for some special classes of graphs there were some polynomial time algorithm found in order to state the optimal number of required colours, the chromatic number $\chi(G)$.

One of these is the class of graphs with prescribed cycle lengths. In 1992 Gyárfás made a first approach on graphs with prescribed odd cycle lengths ([Gyá92]). Mihók and Schiermeyer enlarged in [MS04] his observations and proved analogue statements for graphs with prescribed even cycle lengths. In the end both results were combined in order to provide a polynomial time colouring algorithm, called MAXBIP, for such graphs.

Mainly we will discuss graphs with two given odd cycle lengths. We will find out that for some additional conditions the bound given by Gyárfás can be improved.

This paper starts in chapter 1 with some well known basic definitions and properties of graphs and colourings. Afterwards, in chapter 2 we will present the work of Gyárfás and Mihók and Schiermeyer in particular. We will discuss the main results and their proofs and introduce the algorithm MAXBIP. In the second part of this chapter, we will provide some information about graphs containing only 3- and 5-cycles as odd cycles, done by Wang ([Wan96]). There will also appear a polynomial time colouring algorithm.

Chapter 3 contains the new results done by the author. It will show that 4 colours are sufficient in order to provide a proper colouring of graphs with only 2 consecutive odd cycle lengths greater than 3. It will also contain some structural information on these graphs and provide a short approach how to colour those graphs with 4 colours.

In the last chapter we will discuss sharper bounds and show the appearing effects when the prescribed odd cycle lengths are non consecutive.

1 Preliminaries

1.1 Graphs and cycles

In this first section we will present some basic definitions and observations on graphs and cycles in graphs.
We start with the most important structure.

Definition 1.1. A *graph* G is an ordered pair of disjoint sets $(V(G), E(G))$, where $V(G) \neq \emptyset$ is the set of vertices of G and $E(G) \subseteq V(G)^2$ the set of edges of G. The integer $|V(G)|$ is called the *order* of G.

Notation. We write V and E if it is clear to which graph G they refer. Often the order of a graph G is denoted by $n(G)$ or just n.

In this paper we only deal with simple graphs, therefore we have to characterize them.

Definition 1.2. An edge e is called *loop* if both endvertices are equal, e.g. $e = \{x, x\} \in V$. A graph G is called *simple* if G contains no loops and 2 vertices $x, y \in V$ are only connected by at least one edge $\{x, y\} \in E$ in G. If there exits such an edge we call x and y *adjacent*, otherwise we call them *independent*. If x is an endvertex of an edge $\{x, y\} \in E$ we say x and $\{x, y\}$ are *incident*.

Notation. For $\{x, y\}$ we shortly write xy.

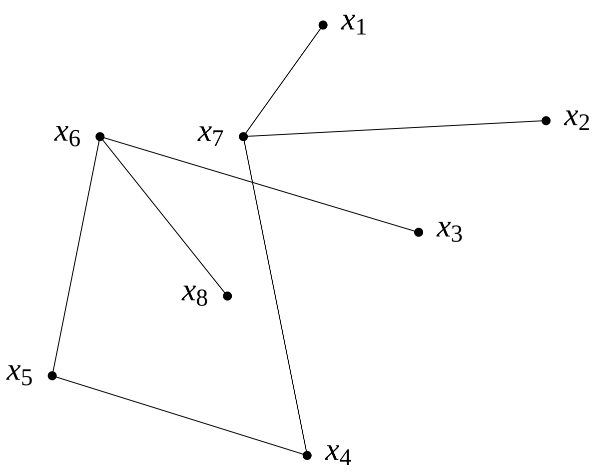

Figure 1.1: Example: $G = (V,E)$ with $V = \{x_1,x_2,x_3,x_4,x_5,x_6,x_6,x_7,x_8\}$ and $E = \{x_1x_7,x_2x_7,x_3x_6,x_4x_5,x_4x_7,x_5x_6,x_6x_8\}$

Definition 1.3. The *neighbourhood* $N(x)$ of a vertex $x \in V$ is the set of all vertices $y \in V$ being adjacent to x. Such a vertex y is called *neighbour* of x.

Definition 1.4. $|N(x)|$ gives the *degree* or *valency* of x. We will denote it by $d(x)$. The *minimum degree* of all vertices of G we will denote by $\delta(G)$ and the *maximum degree* by $\Delta(G)$, respectively. If $\delta(G) = \Delta(G) = k$, that is all vertices of G have valency k, then we call the graph G *k-regular*.

Often we won't handle with the whole graph G, but only with certain substructures.

Definition 1.5. A graph $G' = (V',E')$ is called a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. In this case we write $G' \subseteq G$. If $W \subseteq V$ then we call the graph $(W, W^2 \cap E)$ the *subgraph induced by W*. We denote it by $G[W]$. We call H an *induced subgraph* of G if $H \subseteq G$ and $H = G[V(H)]$.

In our studies the most important substructure is the cycle. We define it now.

Definition 1.6. An alternating sequence of vertices and edges in G, i.e., $x_0e_1x_1 \ldots e_kx_k$, where e_i is incident with x_{i-1} and x_i for all $1 \leq i \leq k$, is called *walk*. If the vertices of this walk are distinct we call it a *path* $P(x_0,x_k)$. The length l of a path/walk is given by its number of vertices.

Definition 1.7. A closed walk, say $x_1 = x_k$, and $k \geq 4$, where all other vertices are distinct from each other and x_1, is called a *cycle* C. An edge $x_i x_j \in E$, where

$$j - i > 1 \mod k \qquad (i < j),$$

we call a *diagonal*.

Notation. Often we mention a walk/way/cycle only by its vertices. For cycles of length k, e.g., $|C| = k$, we use the shortcut C_k, for a path of length k correspondingly P_k.

Definition 1.8. A cycle C_k is called *odd cycle* if k is odd, respectively *even* if k is even.

Definition 1.9. The *distance* of two vertices $x,y \in G$ is the length of a shortest path $P(x,y)$ in G minus 1, denoted by $d(x,y)$.

In the following we define some important structural properties.

Definition 1.10. A graph G is called *connected* if there exits in G for every two vertices $x,y \in V$ a path connecting them. Otherwise the graph is said to be *disconnected*. A maximal connected subgraph in G is called a component of G. A graph G of order at least $k+1$ is said to be *k-connected* if the deletion of every set $S \subset V$ of order at most $k-1$ leads to a connected graph.

Definition 1.11. A vertex set $S \subseteq V$ is said to be *independent* if there exits no edge $xy \in V$ for every two vertices $x,y \in S$. A simple graph G is called *complete* if for every two vertices $x,y \in V$ there exits an edge $xy \in E$.

Remark. For a complete graph G we have $\delta(G) = \Delta(G) = n-1$, where n is the order of G. We denote a complete graph of order n by K_n.

Definition 1.12. A graph G is said to be *p-partite* if we can find a partition of p parts of G, where two vertices are only connected by an edge if and only if both vertices are not members of the same partition class. A 2-partite graph is called *bipartite*.

The following lemma shows us an interesting connection between the occurring cycle lengths in G and the property being bipartite.

Lemma 1.1. *Let G be a simple graph. Then G is bipartite if and only if there are only even cycles in G.*

Proof. Let $G = (V, E)$ be a graph without cycles of odd length. Manifestly a graph is bipartite if all of its components are bipartite. We assume G being connected. Let v be an arbitrary vertex of G. Then we define the two classes $X, Y \subseteq G$ by $x \in X$ iff there exits an even path $P(v, x)$ in G and $y \in Y$ iff there exits an odd path $P(v, y)$ in G. It is easy to see that $v \in X$.
A vertex $u \in V$ can only lie in one set, X or Y, otherwise we could find an even and an odd path from v to u in G. Let $P_e = P(v, u)$ be an even, $P_o = P(v, u)$ be an odd path from v to u. If both paths do not intersect each other, apart from v and u, we easy see that $P_e \cup P_o$ is an od cycle, a contradiction.
If P_e and P_o intersect each other in v_1^1, being the first vertex of the first intersection starting in v, the partial paths P_e^1, being the first part of P_e up to v_1^1, and P_o^1, the first part of P_o up to v_1^1, have to be of same parity, otherwise we could construct an odd cycle using P_e^1 and P_o^1. If $v_1^2 \neq u$ is the last vertex of this first intersection we see that P_e^2, P_e from v_1^2 up to u, and P_o^2, P_o from v_1^2 up to u, must have again different parity.
We can continue this procedure until u is an endvertex of an intersection, leading us to the required contradiction, namely that P_e and P_o must both have the same parity, or until there is no more intersection left. But then we end up in the situation mentioned in the beginning, where P_e and P_o do not intersect. Hence, there cannot exit an odd and an even path from v to u, so every vertex only lies in exactly one set of X and Y, therefore X, Y is a partition of V.

Let x_1, x_2 be two vertices of X. Then there cannot be an edge $x_1x_2 \in E$. Otherwise we could construct an odd path from v to x_1 using the path which brought v_1 into X and the edge x_1x_2 leading to an even and an odd path from v to x_2, a contradiction. With an analogue strategy for Y we show that G is bipartite.

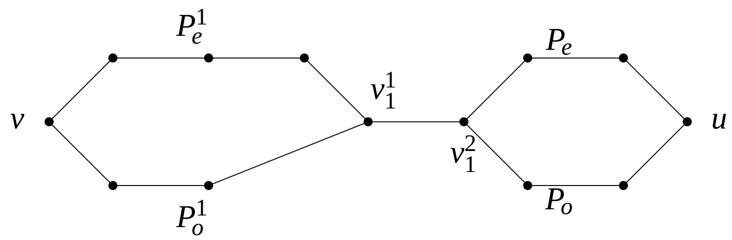

Figure 1.2: Example: $P_e = P_8$, $P_o = P_7$

Assume G is bipartite and we can find an odd cycle $C_k = x_1x_2 \dots x_k$ in G. If both partition classes of G are V_1 and V_2, we know that x_1 has to be an element of exactly one class, say V_1. But then $x_2 \in V_2$, $x_3 \in V_1$, and so on. In the end $x_{k-1} \in V_2$ for $k-1$ being even. Since x_k and x_1 are connected by an edge, x_k has to be in class V_2, a contradiction, for v_{k-1} and x_k cannot be in the same partition class for G being bipartite and $x_{k-1}x_k \in E$. q.e.d.

At last we want to define some special edges and vertices.

Definition 1.13. An edge is called *bridge*, respectively a vertex is called *articulation*, if deleting it would increase the number of components in G.

1.2 About colourings

The first thing we have to know is what a colouring is at all. We will only discuss vertex colourings.

Definition 1.14. A *(vertex) colouring* $\mathscr{C}$ of a graph $G = (V,E)$ is a mapping $\mathscr{C} : V \to S$, where S is the set of the available colours. A *k-colouring* $\mathscr{C}_k$ is a colouring consisting of k colours. A colouring $\mathscr{C}$ is said to be *proper* if every two vertices $x, y \in V$ with $xy \in E$ are coloured with distinct colours. We call a graph G *k-colourable* if we can find a proper k-colouring $\mathscr{C}_k$ of G.

In order to give the best possible colouring of a graph G, say a colouring with as few as possible colours, we define the chromatic number.

Definition 1.15. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k, such that G is k-colourable.

Now we want to give some trivial bounds for $\chi(G)$.

Definition 1.16. A subgraph G' of a graph G is called a *clique* if G' is complete. The order of the largest complete subgraph in G is denoted by $\omega(G)$ and called *clique number of G*.

Lemma 1.2. *Let G be a simple graph. Then*

$$\omega(G) \leq \chi(G) \leq n.$$

Proof. If G is the complete graph K_n then we need n colours for a proper colouring, because every vertex of G is connected by an edge with all other vertices. So we need a distinct colour for every vertex, thus n colours.
According to that we need at least $\omega(G)$ colours to colour the largest clique of G, so at least $\omega(G)$ colours for the whole graph G. q.e.d.

Now we will give a more specific result for an upper bound of $\chi(G)$.

Brooks' Theorem 1.1 ([Bro41]). *Every graph G with $n \geq 3$ is colourable with $\Delta(G)$ colours, apart from cycles with odd length and complete graphs. In these cases we need $\Delta(G)+1$ colours.*

In order to prove this theorem we will present an algorithmic proof by Lovasz [Lov75], simplified by Bryant through his theorem. In the proofs only 2-connected graphs will be observed. If G is only connected then we decompose G into 2-connected parts and colour those. Since recomposing them to G means bonding the parts by a single vertex or an edge, we can expand the colourings to one single colouring.

Bryant's Theorem 1.2 ([Bry96]). *For a graph G the following statements are equivalent:*

1. *G is either the complete graph K_n or a cycle C_n.*
2. *The number of components of G increases after deleting two non-adjacent vertices.*
3. *The number of components of G increases after deleting two non-adjacent vertices connected by a 2-path.*

Proof. The parts 1. $\rightarrow$ 2. and 2. $\rightarrow$ 3. are obvious.
Assume 3. holds and G is not complete. We will show that for every vertex $x \in E$ $d(x) = 2$, and so that G is a cycle.
Since G is 2-connected, but not complete, we know $\delta(G) \geq 2$. Let w be a vertex of G with $d(w) = \Delta(G)$. Then there are two neighbours of w in G, we denote them by u, v, not being adjacent. Otherwise there would be a neighbour of w with its degree greater than $\Delta(G)$. Since G is not complete, but connected, this leads to a contradiction.
Assuming 3. we know there exits a partition of V into non-empty sets $V_1, \{u, v\}, V_2$, such that $w \in V_1$ and after deleting $\{u, v\}$ G splits into V_1 and V_2.
We will show that $V_1 = \{w\}$, leading to $\Delta(G) = 2$.
Assume $|V_1| \geq 2$. Since G is 2-connected, there exits a vertex $x \in V_1 - \{w\}$, w.l.o.g. $xu \in E$. We can assume the same about V_2, so we find a $y \in V_2$ with $uy \in E$. Then with $d(x, y) = 2$ and by 3. G decays after deleting them. So we can find a partition of V,

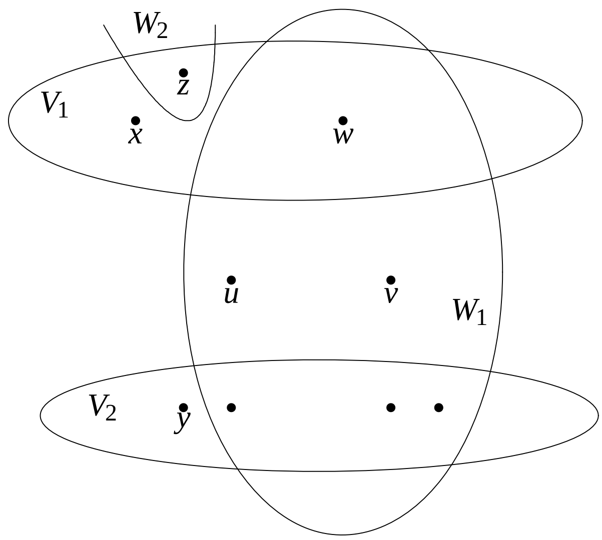

Figure 1.3: Example: $z \in V_1$

say $W_1, \{x,y\}, W_2$. The vertices u,v,w are in one component, assume W_1, and there is a vertex $z \in W_2$.

If $z \in V_1$, respectively V_2, then every path from z to u contains x, respectively y, a contradiction to after deleting either x or y G does not decay, leading us to $\delta(G) = \Delta(G) = 2$. Since G is 2-connected, G has to be a cycle. q.e.d.

Now we can prove *Brooks' Theorem*.

Proof of Theorem 1.1. Consider the following GREEDY-algorithm:

> For a vertex $x \in V$ choose the next smallest available colour out of the colour set $S = \{1,2,3,\dots\}$.

We show that GREEDY can colour G with the colours $1,2,\dots,\Delta(G)$.

Using Theorem 1.2 we know there exist two vertices $u,v \in V$ with $d(u,v) = 2$, such that $G - \{u,v\}$ is still connected. Let w be a vertex of G with $w \in N(u) \cap N(v)$. Denote the vertices of G by $v_1 = u, v_2 = v$ and the remaining ones sequently by non-increasing distances to w.

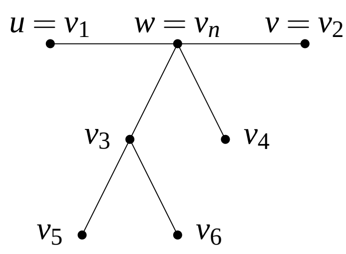

Figure 1.4: Illustration of the used notation.

Now we colour the vertices with GREEDY sequently $v_1, v_2, v_3, \dots$. The notation of the vertices guaranties every vertex v_i with $1 \leq i < n(G)$ being adjacent to a vertex with higher index, which is not coloured yet. Therefore v_i is adjacent to at most $\Delta(G) - 1$ coloured vertices and hence can be coloured with one of the $\Delta(G)$ colours.
In the end $v_n = w$ is adjacent to at most $\Delta(G)$ coloured vertices. Since v_1 and v_2 are both coloured with colour 1, v_n can be coloured with one of the $\Delta(G)$ colours. q.e.d.

Another interesting estimation can be done with the following property.

Definition 1.17. A graph G is called *p-degenerate* if there is no subgraph of G with minimal degree greater than p.

Corollary 1.3. *If G is p-degenerate then $\chi(G) \leq p + 1$.*

Proof. Since G is p-generate we can find a numbering of the vertices of G such that every vertex v_i in this numbering has at most p neighbours in $\{v_1, v_2, \dots, v_{i-1}\}$. Using the GREEDY-algorithm from Theorem 1.1, we can find a proper (p+1)-colouring of G. q.e.d.

An even stronger conjecture was given by Reed in 1999:

Conjecture ([Ree98]). For every graph G with maximum degree $\Delta(G)$ and clique number $\omega(G)$

$$\chi(G) \leq \left\lceil \frac{\omega(G)+1+\Delta(G)}{2} \right\rceil .$$

2 Graphs with prescribed cycle lengths

The aim of this thesis is to provide an upper bound for the chromatic number $\chi(G)$ of a graph G having only odd cycles with two distinct lengths. Before this we want to give a more general bound provided by Gyárfás. Afterwards we will complete this approach with the results of Mihók and Schiermeyer and the analysis of graphs with only 3- and 5-cycles as odd cycles done by Wang.

2.1 Common bounds and colouring algorithm

In 1992 Gyárfás published an interesting result in [Gyá92] containing graphs with k odd cycle lengths. He showed that if a graph G has $k \geq 1$ odd cycle lengths then each block of G contains either a vertex x with $d(x) \leq 2k$ or is a K_{2k+2}. By this he was able to conclude that the chromatic number of G is at most $2k+2$. We will give a short proof of his central theorem, which is based on several lemmata.
For better handling the number of prescribed odd cycle lengths in G we define a special set.

Definition 2.1. Let G be a graph. We define by $C_o(G)$ the set of odd cycle lengths in G, i.e.,

$$C_o(G) := \{2m+1 : G\ contains\ a\ cycle\ of\ length\ 2m+1, for\ m \geq 1\}.$$

With this definition, we know bipartite graphs are graphs with $|C_o(G)| = 0$.

Notation. In the following the graph G will be a 2-connected graph with $\delta(G) \geq 2k+1$ and $|C_o(G)| = k \geq 1$. We will denote a longest odd cycle of G by C. Let G' be the subgraph induced by $V(G) - V(C)$. A longest path of G' will be denoted by P, where a and b will be the endvertices of P.

Theorem 2.1. *If G is a 2-connected graph with $\delta(G) \geq 2k+1$ then $|C_o(G)| = k \geq 1$ implies $G = K_{2k+2}$.*

We need some further information about such graphs in order to give an affirmative proof. We will just state them.

Lemma 2.1. *Let C' be an odd cycle of G'. Then $|C'| < |C|$.*

Lemma 2.2. *If H is the graph we get by adding $2k-1$ diagonals to a cycle T, each diagonal incident to the same vertex $x_0 \in V(T)$, then either H is bipartite or $C_o(H) \geq k$.*

Lemma 2.3. *Let T be a cycle. Let us add $2k-1$ diagonals $e_i = x_0x_i$, $i \in \{1,\dots,2k-1\}$, to T and denote the resulting graph by H. Furthermore assume that H is bipartite and $x,y \in V(H)$, $x \neq y$. Then we can find $k+1$ paths $P^1,\dots,P^{k+1}$ in H from x to y sucht that $|P^i| = |P^j| \mod 2$ holds and $|P^i| \neq |P^j|$ for $1 \leq i < j \leq k+1$.*

Lemma 2.4. *If a is adjacent to p vertices $y_1,\dots,y_p$ of C and to $q+1$ vertices of P and, furthermore, b is adjacent to $y \in V(C) - \{y_1,\dots,y_p\}$ then $|C_o(G)| \geq \lceil p/2 \rceil + q$.*

Lemma 2.5. *If $V(G')$ is an independent set then $G = K_{2k+2}$.*

Lemma 2.6. *If there is a diagonal in P being incident to a and one to b, and, moreover, $N(a) \cap C = N(b) \cap C$, $|N(a) \cap C| = 2k$ then $|C_o(G)| \geq k+1$.*

Lemma 2.7. *$|C_o(G)| \geq k+1$ if $N(a) \cap C = N(b) \cap C$ and $|N(a) \cap C| = 2k$, a and b being distinct.*

Lemma 2.8. *If* $|N(a) \cap C| = 2k$ *and* $y \in (N(b) \cap C)$ *then* $|C_o(G)| \geq k+1$.

Now we can use these lemmata to prove Gyárfás' main result.

Proof of Theorem 2.1. If we know that $a = b$, i.e., there are no edges in G', then Lemma 2.5 implies $G = K_{2k+2}$. So we may assume that a and b are distinct vertices, i.e., $|V(P)| \geq 2$.
If $N(a) \cap C = \emptyset$ (or $N(b) \cap C = \emptyset$) then $d(a) \geq 2k+1$ (or $d(b) \geq 2k+1$) implies that there exits a cycle in G' containing $2k+1$ diagonals incident to the same vertex of this cycle, i.e., a or b. Let us denote this subgraph by H. Applying Lemma 2.2 we know that either H is bipartite or $|C_o(H)| \geq k$. The last case leads with Lemma 2.1 to a contradiction. Hence H is bipartite. From the 2-connectedness of G there exit two vertex-disjoint paths P_1 and P_2 connecting $V(C)$ and $V(H)$. We apply Lemma 2.3 with $x = V(H) \cap P_1$ and $y = V(H) \cap P_2$. Lemma 2.3 ensures $k+1$ paths. Together with P_1 and P_2 and the arc of C, being of suitable parity, the $k+1$ paths ensured by Lemma 2.3 define $k+1$ odd cycles of different lengths. This leads to a contradiction again.
So we can conclude that $N(a) \cap C \neq \emptyset$ and $N(b) \cap C \neq \emptyset$. Using the symmetry of a and b we assume that

$$1 \leq p = |N(a) \cap C| \leq |N(b) \cap C|.$$

Assume $|N(a) \cap P| = q+1$, that means that there are q diagonals of P starting in a. From $d(a) \geq 2k+1$ we get $p+q \geq 2k$.

1. case: $N(a) \cap C \neq N(b) \cap C$.

   By Lemma 2.4 we have

$$|C_o(G)| \geq \left\lceil \frac{p}{2} \right\rceil + q \geq \left\lceil \frac{2k-q}{2} \right\rceil + q = k - \left\lfloor \frac{q}{2} \right\rfloor + q > k,$$

   leading to a contraction, apart from the case that $q = 0$ and $p+q = 2k$. This case is handled in Lemma 2.8. This also leads to $|C_o(G)| \geq k+1$.

2. case: $N(a) \cap C = N(b) \cap C$.

   We can now apply Lemma 2.4 with $p+1$ instead of p and we conclude

$$|C_o(G)| \geq \left\lceil \frac{p-1}{2} \right\rceil + q \geq \left\lceil \frac{2k-q-1}{2} \right\rceil + q = k - \left\lfloor \frac{q+1}{2} \right\rfloor + q > k$$

   apart from the cases $q = 1$ and $p+q = 2k$, or $q = 0$ and $p+q$ equals $2k$ or $2k+1$. Lemma 2.6 treats the case $q = 1$, Lemma 2.7 the case $q = 0$. Both cases lead to $k+1$ odd cycles of distinct lengths, a contradiction. So the only possibility is that $G = K_{2k+2}$, proving the theorem.

q.e.d.

If we have $C_o(G) = k$ then Theorem 2.1 reveals that there exits a proper 2k+1-colouring of each block of G, apart from the block being a K_{2k+2}. Thus the following corollary holds.

Corollary 2.9. *The chromatic number of a graph G with* $|C_o(G)| = k \geq 1$ *is at most 2k+1, unless some block of G being a* K_{2k+2}*. If there exits such a block then* $\chi(G) = 2k+2$.

Unfortunately Theorem 2.1 only provides an upper bound for the chromatic number of such graphs. It gives us no information about the appearance or distribution of these odd cycles. So further structural studies are necessary in order to find a colouring with as few as possible distinct colours.

In [MS04] Mihók and Schiermeyer expanded Gyárfás' approach. They examined the effects of prescribed even cycle lengths. Furthermore they used the polynomial time vertex-colouring algorithm MAXBIP, presented in [MS97], for graphs with prescribed cycle lengths.

In analogy we define the set of even cycle lengths of a graph G.

Definition 2.2. Let G be a graph. Then we denote by $C_e(G)$ the set of even cycle lengths of G, i.e.,

$$C_e(G) = \{2m : G\ contains\ a\ cycle\ of\ length\ 2m,\ for\ m \geq 2\}.$$

Now we can prove an analogue result to Theorem 2.1.

Theorem 2.2. *Let G be a 2-connected graph with $|C_e(G)| = s \geq 1$. Then $\chi(G) \leq 2s+3$. If the equality holds then G contains a K_{2s+3}.*

The theorem assumes that the graph is 2-connected. With the argumentation we used for the Theorems 1.1 and 1.2, we can enlarge the proof being true for all graphs.

Proof of Theorem 2.2. (1) Let $P = v_1v_2 \dots v_p$ be a longest path of order p in G. Then we know $N(v_1) \subseteq V(P)$, otherwise we could extend the path.

Let $I = \{i : v_1v_i \in E(G)\} = A \cup B$, where $A = \left\{a_1, a_2, \dots a_{|A|}\right\}$ contains the odd indices and $B = \left\{2, b_1, b_2, \dots b_{|B|-1}\right\}$ contains the even indices, respectively. We will call A and B the odd and even neighbours of v_1.

If $i, j \in A, i < j$, then the cycle $v_1v_iv_{i+1} \dots v_jv_1$ is of even order $j-i+2$. Whereas, if $i \in B, i \neq 2$, then the cycle $v_1v_2 \dots v_iv_1$ has even order i. Hence $|A| \leq s+1$ and $|B| - 1 \leq s$. Thus we get

$$\delta(G) \leq d(v_1) \leq 1+s+(s+1) = 2s+2. \qquad (*)$$

Since this consideration is true for every subgraph $H \subseteq G$, G is (2s+2)-degenerate, implying $\chi(G) \leq 2s+3$.

(2) If G is a (2s+1)-degenerate graph then $\chi(G) \leq 2s+2$. So, if $\chi(G) = 2s+3$ there has to be a subgraph H of G with $\delta(H) \geq 2s+2$.

Let us consider by $P = v_1v_2\dots v_p$ a longest path in H as in part (1). By $(*)$ we conclude that $d(v_1) = 2s+2 = \delta(H)$. Let us denote by q the index b_1 and among all longest paths in H we choose one such that q is minimal. Considering v_{q-1} the path $P^* = v_{q-1}v_{q-2}\dots v_1v_qv_{q+1}\dots v_p$ is a longest path, too. By the choice of P^*, except for v_{q-2} and v_q, all even neighbours of of v_{q-1} on P^* satisfy $i > q$. Thus there are $s-1$ even indices $i > q$, say $d_1+(q+2), d_2+(q-2), \dots, d_{s-1}+(q-2)$; which leads to $s-1$ even cycles of order $d_1, d_2, \dots, d_{s-1}$. These cycles are given by $v_{q-1}v_qv_{q+1}\dots v_{d_i+q-2}v_{q-1}$. If we replace the edge $v_{q-1}v_q$ by the path $v_{q-1}v_{q-2}\dots v_1v_q$ we get $s-1$ even cycles of order $d_1+(q-2), d_2+(q-2), \dots, d_{s-1}+(q-2)$.

Since there are exactly s even cycle lengths and $q \geq 4$, we conclude that $d_{i-1} = b_{i+1} = d_i+(q+2)$ for $1 \leq i \leq s-2$ and $b_s = d_{s-1}+(q-2)$. Therefore the cycle $v_1v_qv_{q-1}v_{b_2}v_1$ is of order 4 and thus $b_1 = 4 = d_1$, implying $b_i = 2i+2$ for $1 \leq i \leq s$. Necessarily we get $a_{i+1} - a_1 + 2 = b_i$ for $1 \leq i \leq s$.

If $a_1 \geq 5$ the cycle $v_1v_2\dots v_{q-1}v_{a_{s+1}}v_{a_{s+1}-1}\dots v_{a_1}v_1$ has order $(q-1)+(2s+1) = 2s+q = 2s+4 > 2s+2 = b_s$, a contradiction. Hence $a_1 = 3$ implying $a_i = 2i+1$ for $1 \leq i \leq s+1$ and $I = \{2,3,\dots,2s+3\}$. This means that every vertex v_i for $1 \leq i \leq 2s+2$ is an endvertex of a longest path and therefore $G[\{v_1, v_2, \dots, v_{2s+3}\}]$ is isomorphic to the complete graph K_{2s+3} $\forall s \geq 2$.

Finally, let us have a look at the case $s = 1$. If $a_1 < a_2 \leq q-1$ then the cycle $v_1v_{a_1}v_{a_1+1}v_{a_2}v_1$ is of even order $a_2 - a_1 + 2 < q$, a contradiction. Hence $a_2 > q$.

Let d_1, d_2 be the odd neighbours of v_{q-1} on P^*. As seen before we conclude $d_2 > q$. If $a_1, a_2 > q$, or $d_1, d_2 > q$, then there is a cycle of even length greater or equal to $(q-1)+3 = q+2 > q$, a contradiction. If $a_1, d_1 < q-1$ then there is an even cycle of length $|a_1 - d_1| + 1 + 2 + 1 \leq q-2$, again a contradiction. Thus $v_1v_{q-1} \in E(H)$.

Now let us consider the vertex v_{q-2}, which is the endvertex of a longest path, too. As seen above we have $v_{q-2}v_q \in E(H)$ and the cycle $v_1v_{q-1}v_{q-2}v_qv_1$ has order 4. So $q = 4$ and and $G[\{v_1, v_2, v_3, v_4\}]$ is a complete graph of order 4. If $v_iv_j \in E(G)$ for every $v_j \in V(G) - \{v_1, v_2, v_3, v_4\}$ and $1 \leq i \leq 4$ then the complete graph K_5 is a subgraph of G. Otherwise, implied by the 2-connectedness of G, there is an edge $uv \in E(G)$ with $u, v \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that u and v are connected by two vertex disjoint paths with the K_4. But then there must also be an even cycle of length greater or equal to $6 > q = 4$, a contradiction.

q.e.d.

Combining both theorems we get a nice estimation for $\chi(G)$:

Corollary 2.10. *Let G be a 2-connected graph with $|C_o(G)| = k$ and $|C_e(G)| = s$. Then the chromatic number $\chi(G) \leq \min\{2k+2, 2s+3\} \leq k+s+2$.*

Mihók and Schiermeyer even gave a polynomial time vertex-colouring algorithm called MAXBIP, which finds a proper colouring of a given 2-connected graph G satisfying the estimation given in Corollary 2.10.

Algorithm MAXBIP

INPUT a 2-connected graph G

STEP 1 Choose an arbitrary vertex $x_1 \in V(G)$ and add successively vertices $x_2, x_3, \ldots$ to obtain a connected maximal bipartite subgraph $G[V(B_1)]$. Colour the vertices of B_1 properly with colours 1 and 2.

Let $S := V(B_1)$ and $T := N(B_1) - S$

STEP 2 Successively place every vertex of T in the smallest B_i such that $G[V(B_i)]$ is bipartite. 2-colour the vertices of every B_i with the smallest available pair of colours, say $2q-1, 2q$, as done with GREEDY.

Let $S := V(B_1) \cup N(B_1)$ and $R := V(G) - S$.

STEP 3 If $R = \emptyset$ then STOP.

If $R \neq \emptyset$ then let j be the smallest integer such that $N(V(B_j)) - S \neq \emptyset$.

Extend the components of $G[V(B_j)]$ by successively adding vertices to obtain a maximal bipartite subgraph B_j^* in $G[V(B_j) \cup (N(V(B_j)) - S)]$ and extend the 2-colouring of B_j to a 2-colouring of B_j^*.

Set $B_j := B_j^*$.

Let B_j play the role of B_1 in STEP 1.

Let $S := S \cup V(B_j)$ and $T := N(B_j) - S$.

Place successively every vertex of T in the smallest $B_i, i \geq j$, such that $G[V(B_j)]$ is bipartite. Extend the given 2-colourings, again.

Let $S := S \cup N(B_j)$ and $R := V(G) - S$ and repeat STEP 3.

OUTPUT $B_1, B_2, \ldots, B_m,\ m \geq 1$

2.2 Graphs with $C_o(G) = \{3,5\}$

In the next sections we only deal with given odd cycle lengths. In particular we want to study graphs with only two distinct odd cycle lengths. In [Wan96] Wang investigated such graphs. She gave a complete characterization of the case $C_o(G) = \{3,5\}$. We want to recall her results in this section.

First we give a lemma, which will be as basic as the answer, why we only discuss 2-connected graphs, for the next chapters.

Lemma 2.11. *Let G^* be the graph obtained by iteratively deleting all vertices of degree less than i from G. Then for all $i \leq k$ if G^* is k-colourable G is k-colourable, too.*

Proof. If we have coloured G^* with k colours then we can add all deleted vertices in reverse order and colour them with one of the k colours, since these vertices have less then i neighbours in G, so less then $i \leq k$ colours are already used for its neighbours, hence, at least one colour has to be left over for creating a proper colouring of G. q.e.d.

The consequence of this lemma is, that we can now limit our focus to graphs with minimal degree at least 3, since in our requested graphs is at least one odd cycle. So we need 3 colours or even more to obtain a proper colouring.

Wang was interested in graphs G with 3- and 5-cycles being the only odd cycles in G. She formulated an algorithm with complexity $O(E(G))$. We will just state the underlining characterization of such graphs and give a proof.

Theorem 2.3. *Let G be a 2-connected graph with $C_o(G) = \{3,5\}$. Then*

1. $\chi(G) = 6$ *if G contains a K_6,*
2. $\chi(G) = 5$ *if G contains a K_5, but no K_6,*
3. $\chi(G) = 4$ *if G contains a K_4, but no K_5,*
4. $\chi(G) = 4$ *if G contains a W_5, but no K_4, and*
5. $\chi(G) = 3$ *if G contains no W_5 and no K_4.*

A W_5 is a so-called wheel of order 6, where all vertices of a 5-cycle are adjacent to one center vertex.

Proof. From Lemma 2.11 we know that we only need to observe graphs with minimum degree at least 3.
We now consider 3 cases:

case 1 : G contains either a K_6 or G contains a K_5, but no K_6.

Then it is clear that G contains at most 6 vertices. We know from Theorem 2.1 that K_6 is 6-colourable. If G contains a K_5, but no K_6, then G is obviously 5-colourable.

case 2 : G contains a K_4, but no K_5.

Then we can find a clique of order 4 in G. We call it K_4 and make the following observations:

2.1. All vertices in $G - K_4$ are distance one away from from K_4. Otherwise, if $x \in V(G - K_4)$ is a vertex being away from K_4 with distance at least 2 then we can find two distinct paths $P(x, x_1)$ and $P(x, x_2)$ of length at least 3 with $x_1, x_2 \in V(K_4)$. So we could identify an odd cycle with length greater than 5, a contradiction.

2.2 $G - K_4$ does not contain paths of length three. Otherwise, if yxz is a path in $G - K_4$ we could construct an odd cycle of length greater than 5 as done in the last observation.

From observation 2.1. we know that every vertex of $G - K_4$ is adjacent to at least one vertex of K_4. Observation 2.2. tells us that each connected component of $G - K_4$ is either an independent vertex or a path of length 2. Note that each vertex in $G - K_4$ is adjacent to at most three vertices of K_4 for G not containing a K_5. Due to the degree of G being at least 3, an independent vertex of $G - K_4$ is adjacent to exactly 3 vertices of K_4. A vertex of a path of length 2 in $G - K_4$ is adjacent to 2 or 3 vertices of K_4. Because of this we get that $G - K_4$ can not contain a path of length 2 and an independent vertex, for otherwise we could find a 7-cycle. Hence, the connected components of $G - K_4$ are either all paths of length 2 or all independent vertices.

If there are only independent vertices, then all of them have to be adjacent to the same three vertices of K_4, due to the last argument. If there are only paths of length 2 then all paths are adjacent to the same two vertices of K_4, too. Otherwise we could again construct a 7-cycle.

Knowing this a 4-colouring is very easy to establish by colouring K_4 with four colours and then colouring the vertices of $G - K_4$ with one of these for them being only adjacent to at most 3 other already coloured vertices (application of Lemma 2.11).

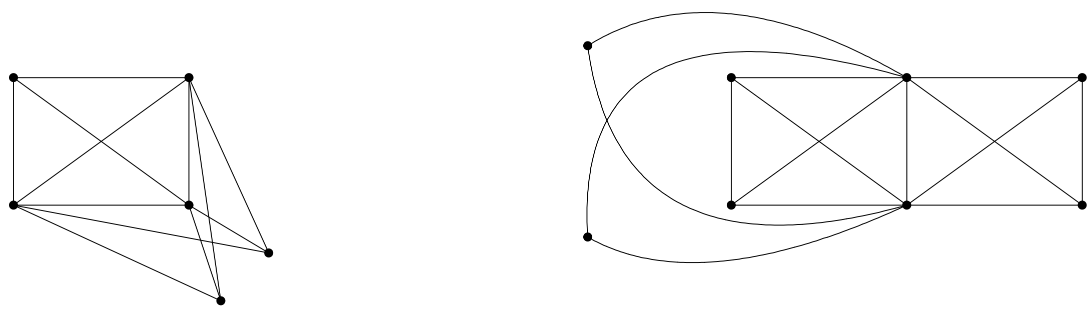

Figure 2.1: Two families of graphs containing a K_4 but no K_5 and only 3-cycles and 5-cycles as odd cycles.

case 3 : G contains no K_4.

We oberserve the following things

3.1. If G contains a W_5 as a subgraph then G is isomorphic to W_5. Otherwise, if there exits a vertex $x \notin W_5$ then there have to be at least 2 different connections between x and W_5, but this leads to an odd cycle of length at least 7, a contradiction.

3.2. Let us consider a 3-cycle. Let r be a vertex of it. Then let us denote the neighbourhood of r by $N_1(r)$, the neighbourhood of $N_1(r)$ by $N_2(r)$ and so on. Then we obtain that there cannot be a 5-cycle in one $N_i(r)$, otherwise we would find a W_5 or a an odd cycle greater than a 5-cycle. Furthermore, there cannot be a 3-cycle in any neighbourhood $N_i(r)$, otherwise we could identify a K_4 or a cycle of odd length greater than 5. So $N_i(r)$ is bipartite for every $i > 0$. Anymore, there are no edges in $N_i(r)$, for $i \geq 3$ for them leading to odd cycles with length greater than 5. We can also obtain that there are no edges between two vertices of $N_2(r)$ if these two vertices share neighbours in $N_1(r)$ being joined by an edge. Otherwise there would be a K_4. (Recall by the construction of the long odd cycles that G is 2-connected.)

Hence we can colour r with colour 1, $N_1(r)$ with the colours 2 and 3, $N_2(r)$ with the colours 1 and 2, $N_3(r)$ with colour 3, $N_4(r)$ with colour 1 and so on.

q.e.d.

An interesting observation is that $N_4(r)$ is empty. If we have a look at the starting 3-cycle in 3.2. then there has to be a path from a vertex in $N_4(r)$ to one of the two other vertices of the 3-cycle, not using r. But then we can easily find an odd cycle with length greater than 5.
Using this last observation, Wang formulated her $O(E(G))$ algorithm:

algorithm

INPUT a 2-connected, non-bipartite graph G with $\delta(G) \geq 3$ and $C_o(G) = \{3,5\}$

STEP 1 If G contains no more than six vertices then colour G using $\chi(G)$ colours. STOP

STEP 2 If G contains no K_5 then

STEP 3 If G contains a K_4 then

STEP 4 Delete (iteratively) all vertices in G with degree three to obtain G^*.

STEP 5 Add all the vertices deleted in STEP 4 back to G^* and 4-colour them. STOP

STEP 6 Else if G contains no K_4 then

STEP 7 Select a vertex r on a 3-cycle to be the root.

STEP 8 Build $N_i(r)$ for $i = 1, 2, 3$.

STEP 9 Colour the root vertex r with colour 1.

STEP 10 Colour the vertices of $N_1(r)$ with the colours 2 and 3.

STEP 11 Colour the vertices of $N_2(r)$ with the colours 1 and 2.

STEP 12 Colour the vertices of $N_3(r)$ with the colour 3.

OUTPUT an optimal colouring of G

3 Graphs with two odd cycles of consecutive lengths

In this chapter we will analyze graphs with only two distinct cycle lengths. They will be of the form $2m-1$ and $2m+1$, thus of consecutive lengths. We will spare the case $m=2$ for being discussed comprehensively in the last chapter. Our aim is to show, that two consecutive odd cycle lengths provide us such a lot information that finally we will be able to colour those graphs with at most 4 colours.

Again we will only discuss 2-connected graphs with minimal degree at least 3 as a consequence from Lemma 2.11 and the argumentation in front of the Theorems 1.1 and 1.2.

We start with some structural information.

3.1 Structural notes

Having only two distinct odd cycle lengths is a very restrictive property. The following lemma shows in which way the odd cycles of a graph G can appear.

Lemma 3.1. *Let G be a 2-connected graph. $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$. Then all C_{2m-1} in G are induced and*

1. *If G contains at least 2 odd cycles of length $2m-1$, say C_1 and C_2, sharing no*

common vertex, then w.l.o.g.

$$x_1y_1 \in E(G)$$

and G contains one of the edges

$$\{x_iy_i : 2 \leq i \leq m\},$$

where $C_1 = x_1x_2 \dots x_{2m-2}x_{2m-1}$ *and* $C_2 = y_1y_2 \dots y_{2m-2}y_{2m-1}$.

2. *There are no two cycles of length* $2m+1$ *not sharing at least one common vertex.*

3. *There are no vertex-disjoint odd cycles of different lengths.*

Proof. Since $C_o(G)$ has only two distinct cycle lengths, there cannot be any diagonal in a C_{2m-1}, because otherwise we would get two new cycles, an odd and an even one. But the odd cycle would have a length less than $2m-1$, a contradiction.

1. It is easy to see that we need at least two distinct paths from $x_i \in C_1$ to $y_j \in C_2$ in order to guarantee the 2-connectedness of G. Assume $x_1y_1 \in E(G)$ and w.l.o.g. let

   $$x_iy_j \in E(G) \text{ with } 2 \leq i \leq m,\ 2 \leq j \leq m,\ i \leq j$$

   be another edge from C_1 to C_2.

   Then we have 4 other cycles than C_1 and C_2:

   1.1. $C_a : x_1\overrightarrow{C_1}x_iy_j\overleftarrow{C_2}y_1$ with length $i+j$,

   1.2. $C_b : x_1\overleftarrow{C_1}x_iy_j\overrightarrow{C_2}y_1$ with length $(2m-1)-(i-2)+(2m-1)-(j-2) = 4m+2-i-j$,

   1.3. $C_c : x_1\overrightarrow{C_1}x_iy_j\overrightarrow{C_2}y_1$ with length $i+(2m-1)-(j-2) = i-j+2m+1$ and

   1.4. $C_d : x_1\overleftarrow{C_1}x_iy_j\overleftarrow{C_2}y_1$ with length $j+(2m-1)-(i-2) = j-i+2m+1$.

   For each case we can find the following estimations:

1.1. $|C_a| = i + j \leq 2m$

1.2. $|C_b| = 4m + 2 - i - j \geq 2m + 2$

1.3. Since $2 - m \leq i - j \leq 0 \Longrightarrow m + 3 \leq |C_c| = i - j + 2m + 1 \leq 2m + 1$.

1.4. Since $0 \leq j - i \leq m - 2 \Longrightarrow 2m + 1 \leq |C_d| = j - i + 2m + 1 \leq 3m - 1$.

From the cycle length of C_b we get i and j must both be even or odd, otherwise we get a contradiction to $2m+1$ being the longest odd cycle length. But in C_c and C_d this would lead to an odd cycle length greater than $2m+1$ apart from the case $i = j$, where C_c and C_d have length $2m+1$.

If we allow instead of edges x_1y_1 and x_iy_j paths $P(x_1, y_1)$ and $P(x_i, y_j)$, then the lengths of the analogue defined cycles $C_a, \ldots, C_d$ would increase at least by 1. One can easy prove that we get an odd cycle in C_b or C_d longer than $2m+1$.

2. An analogue procedure as done in 1. leads to the statement.

3. The procedure can be easily converted into this situation with one difference. The second edge from C_{2m-1} to C_{2m+1} is

$$x_iy_j \text{ with } 2 \leq i \leq m, 2 \leq j \leq m+1.$$

So we get the following lengths

3.1. $|C_a| = i + j \leq 2m + 1$

3.2. $|C_b| = (2m-1) - (i-2) + (2m+1) - (j-2) = 4m + 4 - i - j \geq 2m + 3$

3.3. $|C_c| = i + (2m+1) - (j-2) = i - j + 2m + 3$

3.4. $|C_d| = j + (2m-1) - (i-2) = j - i + 2m + 1$

Applying the conclusion of a) we see that

- if i and j are of different parity, then C_b will be an odd cycle of length greater than $2m+1$, a contradiction.

- Since $(i-j) = -(j-i)$, if i and j are of same parity, then either C_c or C_d have odd length greater than $2m+1$, a contradiction.

With the last statement of a) we finally conclude the assertion.

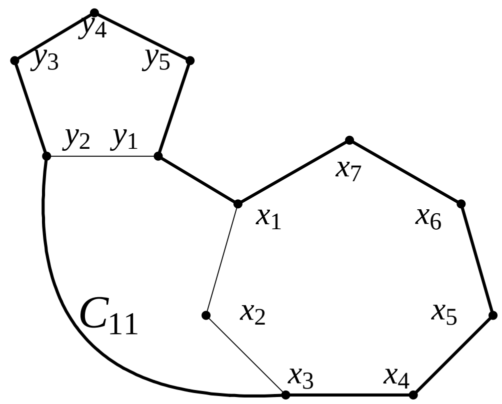

Figure 3.1: Example: $m = 3,\ i = 3,\ j = 2,\ C_d = C_{11}$

q.e.d.

We see that two (2m+1)-cycles have to share at least one vertex. The same is true for two odd cycles of distinct length. Furthermore the appearance of two (2m-1)-cycles is very restricted, too. Let us have a look at the case $m = 3$. The only appearing possibilities restricted by Lemma 3.1 shows the next figure.

Since we limit our view to odd cycles C_{2m-1} and C_{2m+1} with $m \geq 3$, we only discuss graphs with at least 7 vertices. Hence we are able to apply the following theorem by Voss.

Theorem 3.1. *[Vos91] Every non-bipartite 2-connected graph G of order at least 7 and with $\delta(G) \geq 3$ contains odd and even cycles with at least two diagonals.*

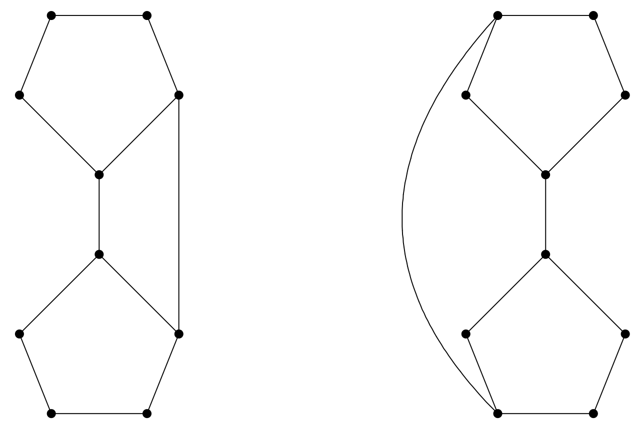

Figure 3.2: Example to Lemma 3.1 with $m = 3$

The consequence of this theorem is that we can find in every graph with $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$ and minimal degree at least 3 an odd cycle of length $2m+1$ with at least 2 diagonals. Let us denote it by C^*_{2m+1}.

Definition 3.1. Let G be a 2-connected graph with minimal degree at least 3 and $C_o(G) = \{2m-1, 2m+1 : m \geq 3\}$. Then we get G' by the induced subgraph of G with $V(G') = V(G) - V(C^*_{2m+1})$.

This notation leads to an interesting observation.

Corollary 3.2. *G' is bipartite.*

Proof. Applying lemma 3.1 we see that every odd cycle in G shares at least one vertex with C^*_{2m+1}. If we delete C^*_{2m+1}, the remaining graph G' cannot contain any odd cycle. Consequently, with Lemma 1.1, G' has to be bipartite. q.e.d.

That is very nice, indeed. If we now 2-colour G' it seems very natural, that colouring C^*_{2m+1} cannot be such a big problem, especially the number of the required colours should be less than the by Theorem 2.1 given 5. We will discuss this in the next section.

Meanwhile we have a closer look at G'.

Lemma 3.3. *Let G be a 2-connected graph with $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$. If there exists a vertex x in G' having three neighbours, say x_1, x_i and x_j, $1 < i < j$, on C^*_{2m+1}, then $P(x_1,x_j)$ is a 5-path on C^*_{2m+1}.*

Proof. Let us first denote some useful cycles to work with:

$$C_u = xx_1\overleftarrow{C^*_{2m+1}}x_j$$
$$C_o = xx_1\overrightarrow{C^*_{2m+1}}x_j.$$

Let x_1, x_i and x_j be the neighbours of x on C^*_{2m+1}. Then we have to study three cases:

1. case: i is even, j is odd

   These assumptions lead us to xx_1x_i and xx_ix_j are both odd cycles, say $C_{2k-1} = xx_1x_i$ and $C_{2n+1} = xx_ix_j$. So $i = 2k$ and $j = 2k+2n-1$. Then C_u is odd with length

   $$|C_u| = (2m+1)-(2k+2n-1)+2+1 = 2m-2k-2n+5.$$

   Since $k,n > 1$ and $i < j$, C_{2k+1}, C_{2n+1} and C_u must have length $2m-1$, so $k = n = m-1$.

   We get

   $$|C_u| = 2m-2k-2n+5 = 2m-2(m-1)-2(m-1)+5 = -2m+9$$
   $$\Rightarrow -2m+9 = 2m-1 \qquad \Rightarrow m = \frac{10}{4},$$

   a contradiction to m being an integer.

2. case: i is even, j is even

   Assume xx_1x_i is a (2k-1)-cycle and xx_ix_j is a (2n)-cycle, then $i = 2k$, $j = 2k-2n-2$.

Now C_u is an even cycle, but C_o is odd with length

$$|C_o| = (2k+2n-2)+1 = 2k+2n-1.$$

Again k must be $m-1$, otherwise $n=1$, a contradiction to G being simple. This leads us to

$$\begin{aligned}|C_o| &= 2k+2n-1 = 2(m-1)+2n-1\\ &= 2m+2n-3.\end{aligned}$$

If C_o is of length $2m-1$, then

$$2m+2n-3 = 2m-1 \qquad \Rightarrow n = 1,$$

a contradiction. If C_o is of length $2m+1$, we get

$$2m+2n-3 = 2m+1 \qquad \Rightarrow n = 2.$$

3. case: i is odd, j is odd

   In this case we get two even cycles. We call xx_1x_i a C_{2k} and xx_ix_j a C_{2n}, so $i = 2k-1$ and $j = (2k-1)+(2n-1)-1 = 2k+2n-3$. This leads to a C_u with length

   $$|C_u| = (2m+1)-(2k+2n-3)+1+2 = 2m-2k-2n+7.$$

   Lets have a closer look at $C_r = xx_j\overrightarrow{C^*_{2m+1}}x_i$. This cycle has length $|C_r| = (2m+1)-(2n-1)+2+1 = 2m-2n+5$. Two cases can appear

   3.1. $|C_r| = 2m-1$

   Then we get

   $$|C_r| = 2m-2n+5 = 2m-1 \qquad \Rightarrow n = 3,$$

leading to

$$
\begin{aligned}
|C_u| = 2m-2k-2n+7 = 2m-2k-6+7 &= 2m-2k+1 \\
&\Rightarrow 2m-2k+1 = 2m+1 \qquad \Rightarrow k=0 \\
&\Rightarrow 2m-2k+1 = 2m-1 \qquad \Rightarrow k=1,
\end{aligned}
$$

which are both a contradiction.

3.2. $|C_r| = 2m+1$

We have

$$|C_r| = 2m-2n+5 = 2m+1 \qquad \Rightarrow n=2,$$

leading to

$$
\begin{aligned}
|C_u| = 2m-2k-2n+7 = 2m-2k-4+7 &= 2m-2k+3 \\
&\Rightarrow 2m-2k+3 = 2m+1 \qquad \Rightarrow k=1,
\end{aligned}
$$

a contradiction

$$\Rightarrow 2m-2k+3 = 2m-1 \qquad \Rightarrow k=2.$$

It is easy to verify, that the case i is odd, j is even, can be reduced to the first case. Beyond that the last two cases are identical if we start to relabel the vertices of C^*_{2m+1} beginning at x_i as the new x_1, keeping the old sequence of labeling. Thus we can w.l.o.g. confirm $i=3$ and $j=5$, stating the lemma. q.e.d.

This lemma provides us the example shown in Figure 3.3.

Furthermore, the lemma tells us about the number of possible neighbours of vertices of G' on C^*_{2m+1}.

Corollary 3.4. *A vertex $x \in V(G')$ can be adjacent to at most three vertices on C^*_{2m+1}.*

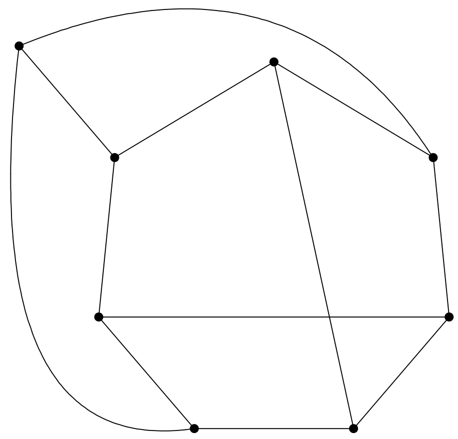

Figure 3.3: Example for $m = 3$ and $\delta(G) = \Delta(G) = 3$.

Proof. From Lemma 3.3 we know the structure of the appearance of a vertex $x \in V(G')$ having at least three neighbours on C^*_{2m+1}. If there are actually four neighbours, every choice of three of them has to fulfill the stated properties.
Let x_1, x_i, x_j, x_l be the neighbours of x on C^*_{2m+1}. Then we know from Lemma 3.3, that we can set $i = 3$ and $j = 5$. For the choice of x_i, x_j, x_l as neighbours used in the lemma, we get with our last notation $l = 7$. If we now choose x_1, x_i and x_l for Lemma 3.3, we get a contradiction. q.e.d.

Until now we have discussed vertices of G' having exactly 3 neighbours on C^*_{2m+1}. But what situations are possible for vertices having only 2 neighbours, especially can they be connected in G'? The next lemma gives an answer.

Lemma 3.5. *Let G be a 2-connected graph with* $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$. *If there exist two vertices* $x^1, x^2 \in V(G')$ *with* $|C^*_{2m+1} \cap N(x^i)| \geq 2$ *for* $i = 1,2$, *then* x^1 *and* x^2 *are not adjacent.*

Proof. Assume x^1 and x^2 are adjacent. It is sufficient to show the lemma for x^1, x^2 having exactly two neighbours.

If x^1 is adjacent to 2 distinct vertices x_i, x_j $(i < j)$ on C^*_{2m+1}, then there exit two cycles $C^1 = x^1 x_i \overrightarrow{C^*_{2m+1}} x_j$ and $C^2 = x x_i \overleftarrow{C^*_{2m+1}} x_j$, where their lengths are of different lengths:

- $|C^1| = j - i + 2$
- $|C^2| = (2m+1) - (j-i) + 2 = 2m - (j-i) + 3$,

if $(j-i)$ is odd, C^1 is odd and C^2 is even, and reverse.
Assume x_i, x_j are labeled in such a way, that C^1 is odd. Assume furthermore $i = 1$, then it is easy to see that j is $2m$ or $2m-2$, leading to C^2 being of length 4 or 6. Let the 2 neighbours of x^2 on C^*_{2m+1} be labeled by x_k and x_l with $(k < l)$. We will now examine the location of x_k and x_l on C^*_{2m+1} in respect to C^1.

1. case: $i \leq k < l \leq j$

   If $i = k$ or $l = j$, then x^1 and x^2 cannot be connected by an edge. Otherwise we would find a triangle. If we assume the cycle $C = x^2 x_k x_l$ being an odd one, we see $k = 2$ and $l = j - 1$. But this leads to a contradiction, because the cycle $x_1 x^1 x^2 x_2 \overrightarrow{C^*_{2m+1}} x_1$ would be an odd cycle with length $2m+3$.

   Let us assume C being even. Then either the path $x_1 \overrightarrow{C^*_{2m+1}} x_k$ or $x_l \overrightarrow{C^*_{2m+1}} x_j$ is even. W.l.o.g let $x_1 \overrightarrow{C^*_{2m+1}} x_k$ be the even path. As we have seen before, k has to be greater than 2. So we get an odd cycle by $x^1 x^2 x_l \overrightarrow{C^*_{2m+1}} x_j$ with length at most $2m-1$. We only get length $2m-1$ if C is a 4-cycle, C^1 is an (2m+1)-cycle and $k = 2$, a contradiction as we have seen before.

2. case: $i < k, j < l$

   It is easy to verify that C has to be a 6-cycle. Otherwise either $l = j+1$ or $l = j+3$, leading to a contradiction as we have seen earlier. (We leave the case $k = j$, leading to a triangle as seen in the beginning of the lemma.) So C can only be a 6-cycle. We get

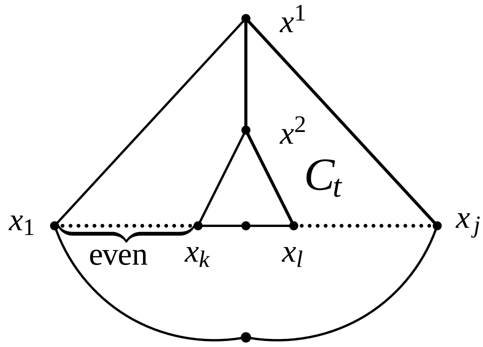

Figure 3.4: Example: t is odd, $j = 2m$

2.1. $l = j+1$ or $k = j-1$: The contradiction is well-known by now.

2.2. $l = j+2$: We can find a 5-cycle, namely $x^1x^2x_lx_{j+1}x_j$ implying $2m+1 = 7$ and $2m-1 = 5$. Then we get $j = 4, l = 6, k = 2$, leading to a 9-cycle, namely $x_1x^1x^2x_2x_3x_4x_5x_6x_7$, a contradiction.

q.e.d.

This leads directly to a nice observation on possible colourings.

Lemma 3.6. *Let G be a 2-connected graph with $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$. If G' is connected then two consecutive vertices x and y on C^*_{2m+1} can only have similar coloured neighbours in G'.*

Proof. Assume there are x_1, being a neighbour of x in G', and y_1, neighbour of y in G', with diffenrent colours in a proper colouring $\mathscr{C}$ of G'. Since G' is bipartite and connected, there has to be an even path $P(x_1,y_1)$ in G'. But then we can construct an odd cycle

$$C = xx_1P(x_1,y_1)y_1y\overrightarrow{C^*_{2m+1}}x$$

with length at least $2m+3$, a contradiction. q.e.d.

Another interesting thing to know is if G is 2-connected G' will remain at least connected. Unfortunately the next lemma negates this assumption.

Lemma 3.7. *Let G be a 2-connected graph with $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$ and valency at least 3. Let x_1, x_2, x_3, x_4 be four sequenced vertices on C^*_{2m+1}. Then if $N(x_i) \cap G' \neq \emptyset$, for all $i \in \{1,2,3,4\}$, G' is not connected. Consequently G' is not connected for $m \geq 5$.*

Proof. Let us denote an arbitrary neighbour of x_i in G' by y_i, for $i = 1, \ldots, 4$. Assume G' is connected, then by Lemma 3.6 the neighbours of the x_i have to be coloured with the same colour in every proper 2-colouring $\mathscr{C}$ of G'. We get the following cases:

1. case: $y_1 = y_2 = y_3 = y_4$.

   This is not possible for finding a triangle, e.g. $x_1 y_1 x_2$.

2. case: Three vertices of the x_i share a common neighbour.

   Again we can find a triangle as done before.

3. case: Two vertices x_i, x_j share a common neighbour.

   It is clear that $j \neq i+1$, for finding a triangle $x_i y_i x_j$. For $i = 1$ and $j = 3$, it is possible that $y_2 = y_4$, but then we find an odd cycle

   $$C = x_1 y_1 x_3 x_2 y_2 x_4 \overrightarrow{C^*_{2m+1}} x_1$$

   with length $2m+3$, a contradiction. So $y_2 \neq y_4$. But since G' is connected, there must be a connection, an odd path $P(y_2, y_4)$ in G', because y_2 and y_4 are similar coloured. But in this case we can construct

   $$C = x_1 y_1 x_3 x_2 y_2 P(y_2, y_4) y_4 x_4 \overrightarrow{C^*_{2m+1}} x_1,$$

   being an odd cycle of length at least $2m+5$, again a contradiction.

For the case that $P(y_2,y_4)$ contains y_1, this path has to be at least of length five to fulfill the assumptions, so

$$C = x_1x_2y_2P(y_2,y_4)x_4\overrightarrow{C^*_{2m+1}}x_1$$

also leads us to the required contradiction.

So we assume that $y_1 = y_4$. Since G' is free of triangles, $y_2 \neq y_3$. y_1 has to be connected with y_3 by an odd path $P(y_1,y_3)$ in G', leading to an odd cycle

$$C = x_1y_1P(y_1,y_3)y_3x_3x_4\overrightarrow{C^*_{2m+1}}x_1$$

with length at least $2m+3$, a contradiction.

4. case: All y_i are distinct.

   Then we can find an odd path $P(y_1,y_3)$ in G', which gives us an odd cycle

$$C = x_1y_1P(y_1,y_3)y_3x_3x_4\overrightarrow{C^*_{2m+1}}x_1$$

   with length at least $2m+3$, a contradiction.

For $m \geq 5$ we know C^*_{2m+1} is of length 11. When we take 3 diagonals as a basis, the appearance of these diagonals will be described in the second remark to Theorem 3.2, we still have five vertices left, whose valency needed to be satisfied. These vertices are consecutive on C^*_{2m+1} and so fulfill the assumptions of the first part. Therefore G' will not be connected.

q.e.d.

We see, if we have to ”many” vertices in G' they cannot be connected. But the remaining question is, what means ”many”?

3.2 Colouring

As mentioned before, Theorem 2.1 tells us that we need at most 5 colours to provide a proper colouring of our graphs. Since we have odd cycles, we need at least 3 colours. So, in which cases do we need 3, 4 or 5 colours? This section will give us a partial answer to that question.

In order to give a much better bound for the required number of colours, we start with an analysis and a refinement of Lemma 3.1

Lemma 3.8. *Let G be a 2-connected graph.* $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$ *and* $\delta(G) \geq 3$. *Then there are no odd cycles* C_{2m+1} *in G, such that* $|C_{2m+1} \cap C^*_{2m+1}| = 1$.

Proof. Assume we can find an odd cycle C_{2m+1}, such that $|C_{2m+1} \cap C^*_{2m+1}| = 1$. We denote $C^*_{2m+1} = x_1x_2\dots x_{2m+1}$ and the $C_{2m+1} = x_1y_2\dots y_{2m+1}$.
Since G is 2-connected, there must be a second connection between the two odd cycles apart from x_1. Assume there is an edge y_jx_i in G. Then we investigate the following cycles:

1. $C^1 = y_j\overleftarrow{C_{2m+1}}x_1\overrightarrow{C^*_{2m+1}}x_i$
2. $C^2 = y_j\overrightarrow{C_{2m+1}}x_1\overleftarrow{C^*_{2m+1}}x_i$
3. $C^3 = y_j\overleftarrow{C_{2m+1}}x_1\overleftarrow{C^*_{2m+1}}x_i$
4. $C^4 = y_j\overrightarrow{C_{2m+1}}x_1\overrightarrow{C^*_{2m+1}}x_i$.

Now have a look at their lengths

1. $|C^1| = i+j-1$
2. $|C^2| = (2m+1)-i+2+(2m+1)-j+1 = 4m+5-(i+j)$
3. $|C^3| = (j-1)+(2m+1)-i+2 = 2m+2-(i-j)$

4. $|C^4| = (2m+1) - j + 1 + i = 2m + 2 + (i - j)$.

It is easy to see, that for i, j being of the same parity, C^1 and C^2 will be odd cycles. We have to determine two cases:

1. case: $|C^2| = 2m+1$

$$\begin{aligned} 4m+5-(i+j) &= 2m+1 \qquad |+(i+j), -(2m+1) \\ 2m+4 &= (i+j) \\ \Rightarrow |C^1| &= i+j-1 = 2m+4-1 = 2m+3, \end{aligned}$$

   a contradiction with $2m+1$ being the greatest length for an odd cycle in G.

2. case: $|C^2| = 2m-1$

$$\begin{aligned} 4m+5-(i+j) &= 2m-1 \qquad |+(i+j), -(2m-1) \\ 2m+6 &= (i+j) \\ \Rightarrow |C^1| &= i+j-1 = 2m+6-1 = 2m+5, \end{aligned}$$

   again a contradiction.

In the case of different parities of i and j, we have to take a look at the other two cycles:

1. case: $|C^3| = 2m+1$

$$\begin{aligned} 2m+2-(i-j) &= 2m+1 \qquad |+(i-j), -(2m+1) \\ 1 &= (i-j) \\ \Rightarrow |C^4| = 2m+2+(i-j) &= 2m+2+1 = 2m+3, \end{aligned}$$

   a contradiction.

2. case: $|C^3| = 2m-1$

$$\begin{aligned} 2m+2-(i-j) &= 2m-1 \qquad |+(i-j), -(2m-1) \\ 3 &= (i-j) \\ \Rightarrow |C^4| &= 2m+2+(i+j) = 2m+5, \end{aligned}$$

a contradiction.

We see, for every choice of i and j we get an odd cycle longer than $2m+1$.

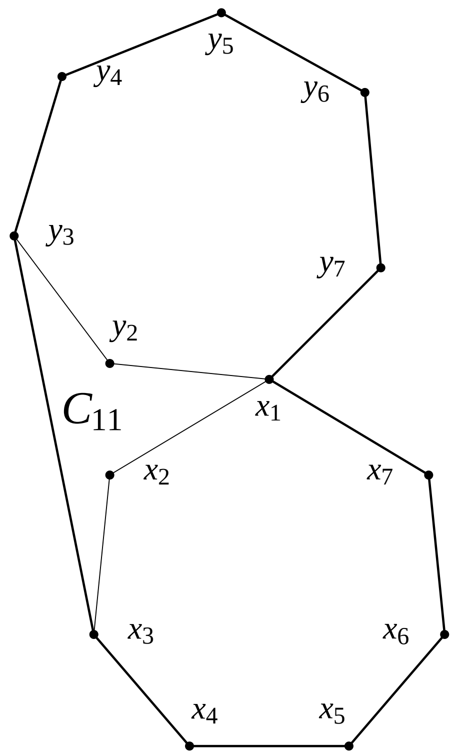

Figure 3.5: Example to Lemma 3.8: $m = 3$, $i = 3$, $j = 3$, $C^2 = C_{11}$

If there are only paths, such as $P(y_j, x_i)$, we can reuse the cycles C^1, C^2, C^3 and C^4 and get analogue contradictions. q.e.d.

We see, two appearing (2m+1)-cycles share at least 2 vertices. We will keep this in mind.

Let us now focus on our special C^*_{2m+1}. If we start with a 2-colouring of G', as mentioned in the last section, we need more information about the connection of C^*_{2m+1} and G'.

Lemma 3.9. *Let G be a 2-connected graph with* $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$ *and* $\delta(G) \geq 3$. *Let x be an arbitrary vertex of* C^*_{2m+1} *such that there exits an even path* $P(x_1, x_{2k})$ *with* $x_1, x_{2k} \in V(N(x) \cap G')$ *(2k gives the length of P). Then for every neighbour y of x on* C^*_{2m+1} *there cannot exit a path in* G' *connecting* $P(x_1, x_{2k})$ *and y. Even so y is not adjacent to an* $x_i \in P(x_1, x_{2k})$.

Proof. The given path can only have length $2(m-1)$, otherwise the occurring cycle $C = xx_1P(x_1, x_{2k})x_{2k}x$ would be an C_{2m+1}, a contradiction to Lemma 3.8, or an odd cycle with length being different from $2m-1$ or $2m+1$, in contradiction with the conditions. Assume there exits an edge yx_i in G', then we can construct two cycles

1. $C^1 = yx_ix_{i+1}\ldots x_{2(m-1)}x\overleftarrow{C^*_{2m+1}}y$
2. $C^2 = yx_ix_{i-1}\ldots x_1x\overleftarrow{C^*_{2m+1}}y$

with lengths

1. $|C^1| = 2(m-1) - i + 1 + (2m+1) = 4m - i$
2. $|C^2| = i + 2m + 1.$

If i is odd, then we have an odd cycle C^1, if i is even, then C^2 is odd. Both odd cycles would be longer than $2m+1$, a contradiction.
If there is no edge, but a path in G', connecting y and $P(x_1, x_{2(m-1)})$, we can reuse both cycles to conclude.

q.e.d.

In order to use as few colours as possible, we need to realize a special starting 2-colouring of G'.

Lemma 3.10. *Let G be a 2-connected graph.* $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$ *and* $\delta(G) \geq 3$. *Then we can find a proper 2-colouring* $\mathscr{C}$ *of* G', *such that an arbitrary* $x \in V(C^*_{2m+1})$ *has only monochrome neighbours in* G' *or its neighbours on* C^*_{2m+1} *have a monochrome neighbourhood in* G'.

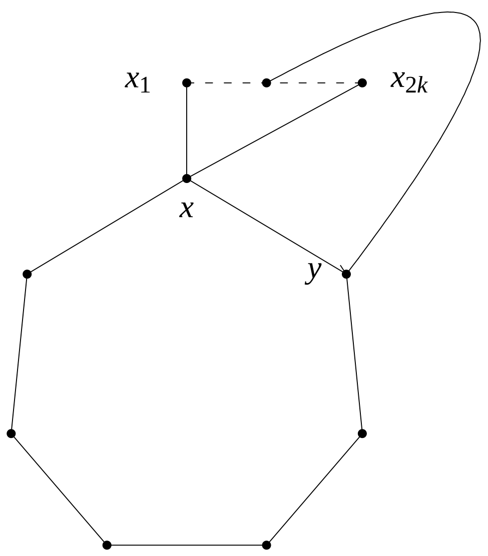

Figure 3.6: Example for Lemma 3.9.

Proof. Assuming we cannot find a proper 2-colouring of G', such that $x \in V(C^*_{2m+1})$ has only monochrome neighbours in G', there have to be two neighbours of x, say y_1 and y_{2k}, being connected by a path $P(y_1, y_{2k})$ of even length in G', because so recolouring of one of these vertices means changing the colour of the other, too.
Lets take this path. We have an odd cycle $C_x = xP(y_1, y_{2k})$. Assume its length is $2m+1$, than we have a contradiction to Lemma 3.8. So its length has to be $2m-1$ and $2k$, giving the length of the connecting path, has to be $2(m-1)$, as stated in Lemma 3.9.
Assume there is no colouring $\mathscr{C}$ of G' so that the right and the left neighbours x_r and x_l of x on C^*_{2m+1} have monochrome neighbourhoods in G'. Then either x_r or x_l or both must have two neighbours connected by an even path in G', or x_y and x_l are in such a way connected, that recolouring the neighbourhood of x_r means recolouring of the neighbourhood of x_l in G'.

1. case: Assume there is an even path in G' connecting two neighbours of x_r in G', x_r^1 and x_r^2, say $P(x_r^1, x_r^2)$. If $|P(x_1, x_{2k}) \cap P(x_r^1, x_r^2)| = 0$ there must be a second

connection (2-connectedness of G) between the both through the paths accrued (2m-1)-cycles. But by Lemma 3.1 this is not possible, because both cycles are already connected by C^*_{2m+1}.

Assume both C_{2m-1} share at least one vertex. By Lemma 3.9 this is impossible, too.

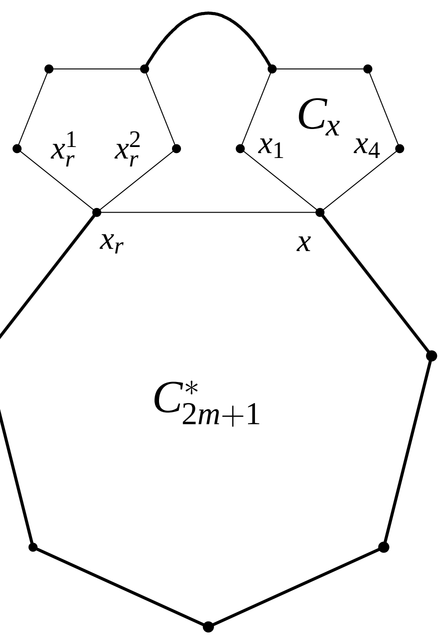

Figure 3.7: Example: $m = 3$, $2k = 4$

Analogue statements are true for x_l.

2. case: Assume x_r and x_l are connected in G' such that recolouring the neighbourhood of one means recolouring the neighbourhood of the other, too. Then, for avoiding the requested 2-colouring, x_r and x_l have to be connected by an odd and an even path simultaneously. There is only one possibility to fulfill this: x_r and x_l share one neighbour y and both are connected by a (2m-4)-path in G'. Shorter or longer paths would create with the C^*_{2m+1} odd cycles of lengths not equal to

$2m-1$ or $2m+1$. The even path could also not be of length $2m-2$, for applying Lemma 3.9 leads to a C_{2m-1} and a C_{2m+1} sharing no vertex, a contradiction to Lemma 3.1.

Assume we can find such a y and a (2m-4)-path. Then we can construct a C_{2m-1} being connected with C_x by the edges $x_r x$ and xx_l. Because of the 2-connectedness of G, we need at least a second connection. This cannot be realized by a path in G', due to Lemma 3.9 again, but also not by a path using vertices of C^*_{2m+1}, a contradiction.

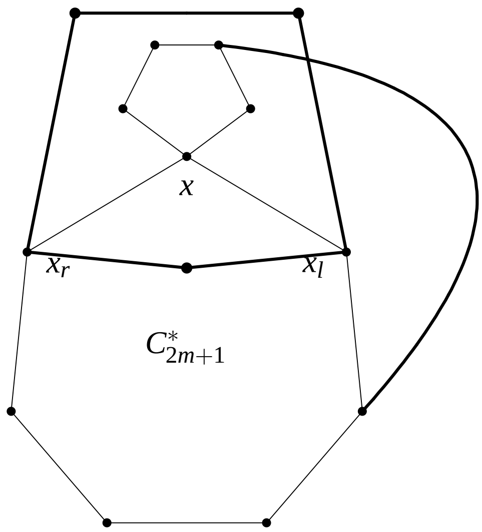

Figure 3.8: Example: $m = 3$

q.e.d.

Now we are able to conclude, that we need at most 4 colours to find a proper colouring of graphs with 2 sequent odd cycle lengths.

Theorem 3.2. *Let G be a 2-connected graph with minimum degree $\delta(G) \geq 3$ and $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$. Then G is 4-colourable.*

Proof. Obtaining Lemma 3.10 we can find a 2-colouring $\mathscr{C}$ of G', where $x \in V(C^*_{2m+1})$ has only monochrome neighbours in G' and $d(x) = 2$ in C^*_{2m+1} or its neighbours on C^*_{2m+1} have only a monochrome neighbourhood in G'.
Then we can either colour x with the colour not used for the its neighbourhood in G' or its neighbours on C^*_{2m+1} can be coloured in the same way and x with a third one.
The rest of C^*_{2m+1} can be coloured alternating with a third and fourth colour. q.e.d.

Remark 1. To the existence of x in Theorem 3.2:
If such an x would not exit then every vertex of C^*_{2m+1} would be incident with a diagonal of C^*_{2m+1}. In accordance with the Pigeonhole Principle there must be a vertex x_1 incident to at least 2 diagonals of C^*_{2m+1}. Without violating the assumptions these are precisely two diagonals: w.l.o.g. x_1x_4 and x_1x_{2m-1}, vertices labeled in appearance on C^*_{2m+1}. But then we can find an odd cycle $C = x_1x_4\overrightarrow{C^*_{2m+1}}x_{2m-1}$ with length $2m-3$, a contradiction.

Remark 2. To the colouring of C^*_{2m+1} in Theorem 3.2.:
Let us denote $C^*_{2m+1} = x_1x_2\ldots x_{2m+1}$ and assume x_1x_i is a diagonal. Then this diagonal divides C^*_{2m+1} into two cycles, an odd one with length $2m-1$ and an even one with length four. Therefore the following conjecture holds:

$$i = \pm 4 \mod 2m+1.$$

Assume $i = 4$. Let x_jx_k be another diagonal of C^*_{2m+1}, w.l.o.g. $j < k$. Then

$$j - k = 4 \mod 2m+1$$

and j has to be 3 or 4. Otherwise we find an odd cycle $C = x_1x_4x_5\ldots x_jx_kx_{k+1}\ldots x_1$ with length $2m-3$, a contradiction. Therefore there can be at most three diagonals in C^*_{2m+1}. Now it is easy to validate that C^*_{2m+1} can be coloured as given in Theorem 3.2.

Remark 3. To the choice of x in Theorem 3.2:
Obviously we have a problem if the chosen x, in the case C^*_{2m+1} contains only two diagonals, is a vertex being present in an even part of C^*_{2m+1}, being split by a diagonal.

Then we have to choose a vertex with degree 2 in C^*_{2m+1} contained in both occurring $2m-1$ cycles on C^*_{2m+1}.

4 Outlook and open problems

As mentioned before the question remains if there are cases such that 3 colours would be sufficient to colour G? The example in figure 3.3. is such a graph. So, what is the speciality of this graph? Or, maybe it is sufficient to use only 3 colours to guaranty a proper colouring?

Problem 1. Let G be a 2-connected graph with minimal degree at least 3 and $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geq 3\}$. Is $\chi(G) = 3$?

Another question is if we do not require from the two odd cycle lengths being consecutive, will we get a similar result as in Theorem 3.2? At least the following lemma shows, that there are some similarities:

Lemma 4.1. *Let G be a 2-connected graph and the set of odd cycle lengths is* $C_o(G) = \{2n+1, 2m+1 : n, m \geq 2, m \geq n+2\}$. *Then 2 odd cycles share at least one common vertex if they are both of length* $2m+1$.

Proof. Assuming there are two C_{2m+1} sharing no common vertex.
Assume there is an edge between the two cycles. We denote both vertices of them by x_1 and y_1 and correspondingly the rest of the vertices of the cycles by $x_2,\ x_3 \ldots$ in accordance with its appearance on the cycle. Finally we call $C^x : x_1 \ldots x_{2m+1}$ and $C^y : y_1 \ldots y_{2m+1}$.
Since G is 2-connected, there has to be a second connection between the two odd cycles. Assume it is a second edge, say $x_i y_j$ with $i > 2,\ j > 1$, and $i, j \leq m+1$. Then we can construct four cycles as in lemma 3.1:

1. $C_a : y_j x_i \overleftarrow{C^x} x_1 y_1 \overrightarrow{C^y} y_j$

2. $C_b : y_j x_i \overrightarrow{C^x} x_1 y_1 \overleftarrow{C^y} y_j$

3. $C_c : y_j x_i \overleftarrow{C^x} x_1 y_1 \overleftarrow{C^y} y_j$

4. $C_d : y_j x_i \overrightarrow{C^x} x_1 y_1 \overrightarrow{C^y} y_j$.

Their lengths are

1. $|C_a| = i + j$

2. $|C_b| = (2m+1) - (i-2) + (2m+1) - (j+2) = 4m+6-(i+j)$

3. $|C_c| = i + (2m+1) - (j-2) = 2m+3+(i-j)$

4. $|C_d| = j + (2m+1) - (i-2) = 2m+3+(j-i) = 2m+3-(i-j)$

If i and j are of different parity, then C_a and C_b are odd cycles. Since $i, j \leq m+1$ we have $|C_b| = 4m+6-(i+j) \geq 4m+-(2m+2) = 2m+4 > 2m+1$ which leads to a contradiction, because the longest odd cycle can only to one of length $2m+1$.

If i and j are of the same parity, then we odd cycles for C_c and C_d. This leads us to 2 cases:

- $|C_c| = 2n+1$:

$$\begin{aligned} 2m+3+(i-j) &= 2n+1 \qquad |-(2m+3) \\ (i-j) &= 2m+2n-2 \\ \Rightarrow 2m3-(i-j) &= 2m+3-2n+2m+2 = 4m-2n+5, \end{aligned}$$

  a contradiction since $m > n$.

- $|C_c| = 2m+1$:

$$\begin{aligned} 2m+3+(i-j) &= 2m+1 \qquad |-(2m+3) \\ (i-j) &= -2 \\ \Rightarrow 2m+3-(i-j) &= 2m+3-(-2) = 2m+5, \end{aligned}$$

  again a contradiction.

All possible odd cycles are of length greater than $2m+1$. So we would get the same result, if the cycles are only connected by paths instead of edges. q.e.d.

But unfortunately we cannot say as in Lemma 3.1, that a (2n+1)- and a (2m+1)-cycle are somehow connected by a common vertex. Furthermore, depending on the difference between *n* and *m* a (2n+1)-cycle and a (2m+1)-cycle can lie very far away from one another. Therefore, we cannot find a central cycle as C^*_{2m+1} in order to guaranty $G\left[V(G) - V(C^*_{2m+1})\right]$ being bipartite.

Problem 2. Let G be a 2-connected graph with minimal degree at least 3 and $C_o(G) = \{2n+1, 2m+1 : n,m \geq 2, m \geq n+2\}$. Is $\chi(G) < 5$?

List of Figures

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Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe.
Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht.

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