Boundary Value Problems for Holomorphic Functions

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1 Introduction

This script is devoted to boundary value problems for holomorphic functions; a living subject with a fascinating history and interesting applications. The main theme is a problem which is nearly as old as function theory itself and can be traced back to Bernhard Riemann’s famous thesis “Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse”. Nowadays is usually called the ‘Riemann-Hilbert problem’ and belongs to the basic problems in complex analysis. Riemann-Hilbert problems have a rich and nice theory, interrelations with several other questions in operator theory, geometric function theory, and approximation theory, and a number of applications in mathematical physics. These lectures are intended as introduction to linear and nonlinear scalar Riemann-Hilbert problems. Not all details and proofs are included; the interested reader is referred to the literature quoted in the text. The plan of the paper is as follows. We start with summarizing relevant results from complex analysis with emphasis on the boundary behavior of analytic functions and continue with a detailed study of linear Riemann-Hilbert problems with continuous and piecewise continuous coefficients. Then we consider several classes of nonlinear Riemann-Hilbert problems. Using fixed points methods, we investigate the existence of solutions to problems with boundary conditions given in explicit form. Finally we present results from the geometric theory of general nonlinear Riemann-Hilbert problems. Among the various applications we sketch two: a problem in hydrodynamics and an optimal design problem for dynamical systems. All boundary value problems are only considered for holomorphic functions in the unit disc, but we point out that everything can be transplanted to arbitrary (smoothly bounded) Jordan domains by conformal mapping. The necessary prerequisites are kept at a minimum, however, it is supposed that the reader is familiar with basic results of functional analysis and classical function theory.

2 Function theory in the disk

In this chapter we recall some properties of holomorphic functions in the complex unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) which are of particular importance for our purposes. The major topics are the existence of boundary values on the complex unit circle \( \mathbb{T} := \partial \mathbb{D} \) and several integral representations of holomorphic functions. Since the material of this section has only auxiliary character, we do not prove the results. Standard references are Duren [4], Garnett [6], Hoffman [12], Koosis [13], Rosenblum and Rovnyak [17], and (for the German reader) Fischer and Lieb [14].
2.1 Holomorphic functions and their boundary values

We denote by \( \mathcal{O}(\mathbb{D}) \) the linear space of holomorphic functions (\( \equiv \) analytic functions) in the unit disk \( \mathbb{D} \). There are several concepts for defining boundary limits of functions in \( \mathcal{O}(\mathbb{D}) \) at a point \( t \) on the unit circle \( \mathbb{T} \):

The unrestricted limit, which does not take advantage of the special behavior of holomorphic functions, is

\[
\lim_{\mathbb{D} \ni z \to t} f(z).
\]

For our purposes this definition is usually too strong. A concept which fits well is the nontangential limit. For \( t \in \mathbb{T} \) and \( 0 < \alpha < \pi/2 \) the set

\[
S(t, \alpha) := \{ z \in \mathbb{D} : |\arg(1 - \overline{t}z)| < \alpha \}
\]

is called a Stolzian angle at \( t \). The function \( f \) is said to have the nontangential or angular limit \( a \) at \( t \in \mathbb{T} \), if

\[
a = \lim_{S(t, \alpha) \ni z \to t} f(z)
\]

for every Stolzian angle at \( t \). We also refer to the angular limit as the boundary value of \( f \) at \( t \) and define \( f(t) := a \). If the boundary values of \( f \) exist almost everywhere on \( \mathbb{T} \) (with respect to the Lebesgue measure) the function \( f^* : \mathbb{T} \to \mathbb{C}, \ t \mapsto f(t) \) is called the boundary function of \( f \). Except in this introduction the boundary function is again denoted by \( f \).

The first fundamental result shows that a holomorphic function is already determined by its angular limits on a set of positive measure.

**Theorem 1.** (Luzin-Privalov). *If \( f \in \mathcal{O}(\mathbb{D}) \) has vanishing nontangential limits on a subset \( T \subset \mathbb{T} \) of positive (Lebesgue-) measure then \( f \equiv 0 \).*

In contrast to this situation there are nontrivial holomorphic function with vanishing radial limit

\[
\lim_{r \to 1^-} f(rt) = 0
\]

almost everywhere on \( \mathbb{T} \), which shows that this concept is, in a sense, too weak.

2.2 Hardy spaces

Since we shall apply functional analytic methods, appropriate spaces of holomorphic functions are required.

Let \( 1 \leq p < \infty \). We say that a holomorphic function \( f \in \mathcal{O}(\mathbb{D}) \) belongs to the Hardy space \( H^p \equiv H^p(\mathbb{D}) \) if

\[
\|f\|_p := \sup_{0 < r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\tau})|^p \, d\tau \right\}^{1/p} < \infty.
\]
If $f \in \mathcal{O}(\mathbb{D})$ is bounded, we write $f \in H^\infty(\mathbb{D})$ and define
\[
\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.
\]
Equipped with the norm $\|\cdot\|_p$ the Hardy spaces $H^p(\mathbb{D})$ are Banach spaces for all $p$ with $1 \leq p \leq \infty$. If $1 < p < q < \infty$ the embeddings $H^\infty \subset H^p \subset H^q \subset H^1$ are continuous.

A famous, and somewhat surprising, result is the existence of a boundary function for all $f \in H^p(\mathbb{D})$. Let $L^p(\mathbb{T})$ denote the Lebesgue spaces on the circle $\mathbb{T}$ with respect to the normalized Lebesgue measure.

**Theorem 2.** (Fatou). Let $1 \leq p \leq \infty$. Any function $f \in H^p(\mathbb{D})$ has a boundary function $f^* \in L^p(\mathbb{T})$. The norms of $f$ in $H^p(\mathbb{D})$ and of $f^*$ in $L^p(\mathbb{T})$ coincide.

By Fatou’s theorem, the embedding $H^p(\mathbb{D}) \to L^p(\mathbb{T})$, $f \mapsto f^*$ is isometric, which shows that $H^p(\mathbb{D})$ can be identified with a closed subspace $H^p(\mathbb{T})$ of functions in $L^p(\mathbb{T})$. In particular, the Hardy space $H^2(\mathbb{T})$, and hence $H^2(\mathbb{D})$, is a Hilbert space.

In what follows we shall frequently not distinguish between the Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ and denote either of the spaces by $H^p$. According to this convention we may also speak of holomorphic functions on the unit circle.

In this connection two natural questions arise which will be answered in the following subsections:

1. Which functions $g \in L^p(\mathbb{T})$ are boundary functions of $f \in H^p(\mathbb{D})$?

2. How functions $f$ in $H^p(\mathbb{D})$ can be recovered from $f^*$ in $H^p(\mathbb{T})$?

In order to answer the first question we associate with any function $g \in L^1(\mathbb{T})$ the sequence of its Fourier coefficients
\[
g_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\tau}) e^{-ik\tau} d\tau, \quad k \in \mathbb{Z}.
\]
If $f$ is holomorphic in $\mathbb{D}$, its *Taylor series*
\[
f(z) = \sum_{k=0}^{\infty} f_k z^k,
\]
converges uniformly on compact subsets of $\mathbb{D}$. The coefficients are given by Cauchy’s integral formula
\[
f_k = \frac{1}{k!} f^{(k)}(0) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t)}{t^{k+1}} dt, \quad 0 < r < 1, \quad k = 0, 1, 2, \ldots .
\]
If this formula is evaluated for $k < 0$, then, by Cauchy’s integral theorem,
\[
f_k = \frac{1}{2\pi i} \int_{\mathbb{T}} f(t) t^{-k-1} dt = 0, \quad k = -1, -2, \ldots.
\]
As \( r \to 1 - 0 \), at least formally, the formulas for the \( f_k \) turn into the formula for the Fourier coefficients of the boundary function \( f^* \)

\[
f_k = \frac{1}{2\pi i} \int_{\mathbb{T}} f^*(t) t^{-(k+1)} \, dt = \frac{1}{2\pi} \int_{0}^{2\pi} f^*(e^{i\tau}) e^{-ik\tau} \, d\tau, \quad k \in \mathbb{Z}.
\]

A rigorous proof of this fact requires a nontrivial result.

**Theorem 3.** (F. and M. Riesz) If \( 1 \leq p < \infty \) and \( f \in H^p \), the functions \( f_r : \mathbb{T} \to \mathbb{R} \) defined by \( f_r(t) := f(rt) \) converge in \( L^p(\mathbb{T}) \) to the boundary function \( f^* \).

Taking this for granted with \( p = 1 \), it shows that the Fourier coefficients \( f_k \) of the boundary function \( f^* \) coincide with the Taylor coefficients of \( f \) if \( k \geq 0 \) and that \( f_k = 0 \) for \( k < 0 \). In fact this condition is not only necessary, but also sufficient:

**Theorem 4.** Let \( 1 \leq p \leq \infty \). Then \( g \in L^p(\mathbb{T}) \) is the boundary function of an \( f \) in \( H^p(\mathbb{D}) \) if and only if the Fourier coefficients \( g_k \) vanish for all negative \( k \). In this case the \( g_k \) are the Taylor coefficients of \( f \) at \( z = 0 \).

Theorem 4 characterizes the space \( H^p(\mathbb{T}) \) of holomorphic functions as the (closed) subspace of functions \( g \in L^p(\mathbb{T}) \) which have vanishing Fourier coefficients \( g_k \) for \( k < 0 \). Accordingly we call those functions \( g \in L^p(\mathbb{T}) \) for which \( g_k = 0 \) for all \( k \geq 0 \) antiholomorphic and denote the corresponding subspace of \( L^p(\mathbb{T}) \) by \( H^p_{\text{a}}(\mathbb{T}) \).

In the Hilbert space \( L^2(\mathbb{T}) \) the functions \( t^k \) form an orthonormal basis and thus \( H^2(\mathbb{T}) \) and \( H^2_{\text{a}}(\mathbb{T}) \) are orthogonal. The orthogonal projection \( P \) of \( L^2(\mathbb{T}) \) onto \( H^2(\mathbb{T}) \) is then also defined on \( L^p(\mathbb{T}) \) if \( p \geq 2 \) and we may ask if it maps (continuously) into \( H^p(\mathbb{T}) \).

For \( 1 \leq p < 2 \) the question is whether or not \( P \) extends continuously to a bounded operator \( P : L^p(\mathbb{T}) \to H^p(\mathbb{T}) \). With the exception of \( p = 1 \) and \( p = \infty \) both questions are answered in the affirmative. The resulting operator \( P \) is called the Riesz projection (or Szegö projection). The complementary projection \( I - P \) is denoted by \( Q \).

**Theorem 5.** If \( 1 < p < \infty \) the Riesz projection \( P : L^p(\mathbb{T}) \to H^p(\mathbb{T}) \) is bounded.

### 2.3 Integral representations

The final topic of this section concerns the reconstruction of holomorphic functions at inner points of \( \mathbb{D} \) from their boundary function on \( \mathbb{T} \). One of several possibilities consists in extending Cauchy’s integral formula to the boundary of the domain.

**Theorem 6.** (Cauchy’s integral formula) If \( f \in H^1(\mathbb{D}) \), then for all \( z \in \mathbb{D} \)

\[
f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(t)}{t - z} \, dt.
\]

The Cauchy integral makes sense for all density functions \( f \in L^p(\mathbb{T}) \) and \( z \in \mathbb{C} \setminus \mathbb{T} \),

\[
\text{Cau} f(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t)}{t - z} \, dt, \quad z \in \mathbb{C} \setminus \mathbb{T}.
\]
The function $Cau f$ is holomorphic in $\mathbb{D}$ as well as in $\mathbb{C} \setminus \overline{\mathbb{D}}$. The boundary behavior of $Cau f$ with an arbitrary density function $f$ for angular limits $z \to t \in \mathbb{T}$ from inside and outside $\mathbb{T}$ is described in the next theorem.

**Theorem 7.** (Plemel’-Sokhotski) If $f \in L^p(\mathbb{T})$ for some $p > 1$, then, in the sense of angular limits,
\[
f_+(t) := \lim_{\mathbb{D} \ni z \to t} Cau f(z) = Pf(t), \quad f_-(t) := \lim_{\mathbb{C} \setminus \mathbb{D} \ni z \to t} Cau f(z) = -Q f(t).
\]

A second possibility for reconstructing holomorphic functions from their boundary functions makes use of the fact that holomorphic functions are also harmonic, and can thus be determined by means of Poisson’s integral formula.

**Theorem 8.** (Poisson’s integral formula). If $f \in H^1(\mathbb{D})$, then for all $z \in \mathbb{D}$
\[
f(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \text{Re} \frac{t + z}{t - z} f^*(t) \frac{dt}{|t|} \equiv \frac{1}{2\pi i} \int_{\mathbb{T}} \text{Re} \frac{t + z}{t - z} f^*(t) \frac{dt}{t}.
\]

The kernel function in the Poisson integral is said to be the Poisson kernel (of the unit disk) and has the alternative representations
\[
K(t, z) := \text{Re} \frac{t + z}{t - z} = \frac{1 - |z|^2}{|t - z|^2} = \frac{1 - r^2}{1 - 2r \cos(\varphi - \tau) + r^2},
\]
with $t = e^{i\tau}$ and $z = re^{i\varphi}$.

In what follows we will frequently use the notation $f(0)$ to denote the mean value of a function $f$ on $\mathbb{T}$. This notation is motivated by Poisson’s integral formula, since the mean value of $f$ coincides with the value $f(0)$ of the harmonic extension.

If the density function is the boundary function of a holomorphic function, the Cauchy and the Poisson formula yield the same result. In order to explain their differences for general densities we study the mapping properties on the monomials $t^k$. The result is

Cauchy: $t^k \mapsto \begin{cases} z^k & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$

for the Cauchy integral formula, while for Poisson’s formula

Poisson: $t^k \mapsto \begin{cases} z^k & \text{if } k \geq 0 \\ \overline{z}^{-k} & \text{if } k < 0 \end{cases}$

This result reflects that Poisson’s integral formula yields a function which is harmonic in $\mathbb{D}$ (but not necessarily holomorphic) and coincides with the density on the boundary, while Cauchy’s integral formula produces a holomorphic function with boundary values that do in general not coincide with the given density function $f^*$. 

6
A third formula for reconstructing a holomorphic function from its boundary values, is based on the fact that a holomorphic function $w$ is essentially determined by the real part of its boundary function: If $w \in H^2$ then

$$w(t) = \sum_{k=0}^{\infty} c_k t^k,$$

where the series converges in $L^2(\mathbb{T})$, and hence

$$\text{Re} w(e^{i\tau}) = \text{Re} c_0 + \sum_{k=1}^{\infty} \text{Re} c_k \cdot \cos k\tau - \sum_{k=1}^{\infty} \text{Im} c_k \cdot \sin k\tau.$$

Consequently, if $\text{Re} w(t) = 0$ almost everywhere on $\mathbb{T}$, then $c_k = 0$ for $k \geq 1$ and $\text{Re} c_0 = 0$. Conversely, if a real-valued function $u \in L^2(\mathbb{T})$ is given by its Fourier series

$$u(e^{i\tau}) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\tau + b_k \sin k\tau),$$

then there exists $w \in H^2$ with $\text{Re} w = u$ on $\mathbb{T}$, namely

$$w(z) = a_0 + \sum_{k=1}^{\infty} (a_k - i b_k) z^k.$$

The function $w$ is unique up to an imaginary constant. A more compact expression for the Schwarz’ operator $u \mapsto w$ is given in the following theorem.

**Theorem 9.** (Schwarz’ integral formula) Let $1 < p < \infty$.

(i) If $w \in H^p(D)$, then for all $z \in D$

$$w(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{t + z}{t - z} \text{Re} w(t) |dt| + i \text{Im} w(0).$$

(ii) If $u \in L^p(\mathbb{T})$ is real-valued, and $w$ is defined in $\mathbb{D}$ by

$$w(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{t + z}{t - z} u(t) |dt|,$$

then $w \in H^p(D)$, $\text{Re} w = u$ almost everywhere on $\mathbb{T}$, and $\text{Im} w(0) = 0$.

Of course, the Schwarz operator also acts on complex valued functions $u$. In particular we have

$$\text{Schwarz: } t^k \mapsto \begin{cases} 2z^k & \text{if } k > 0 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } k < 0, \end{cases}$$
which reveals some similarity with the Cauchy integral.
The real part of the Schwarz kernel \((t + z)/(t - z)\) is the Poisson kernel, already known from Poisson’s integral formula, its imaginary part is said to be the conjugate Poisson kernel. With \(z = re^{i\varphi}\) and \(t = e^{i\tau}\) we have

\[
\begin{align*}
\text{Re} \frac{t + z}{t - z} &= \frac{1 - r^2}{1 - 2r\cos(\tau - \varphi) + r^2}, \\
\text{Im} \frac{t + z}{t - z} &= \frac{2r\sin(\varphi - \tau)}{1 - 2r\cos(\varphi - \tau) + r^2}.
\end{align*}
\]

If \(z\) approaches a point \(e^{i\sigma}\) on the boundary, we obtain the limit

\[
\frac{\sin(\sigma - \tau)}{1 - \cos(\sigma - \tau)} = \cot \frac{\sigma - \tau}{2},
\]

which is said to be the Hilbert kernel. Using this kernel we define the Hilbert operator

\[Hf(e^{i\tau}) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\sigma}) \cot \frac{\sigma - \tau}{2} \, d\sigma,\]

where the integral is to be understood in the Cauchy principal value sense. A straightforward computation (with \(t = e^{i\tau}\)) shows that

\[
H : \begin{cases} 
\sin k\tau &\mapsto \cos k\tau \ (k = 1, 2, 3, \ldots), \\
\cos k\tau &\mapsto -\sin k\tau \ (k = 0, 1, 2, 3, \ldots),
\end{cases}
\]

or, in complex notation,

\[
\text{Hilbert} : \begin{cases} 
-t^k &\text{if} \ k < 0, \\
0 &\text{if} \ k = 0, \\
t^k &\text{if} \ k > 0.
\end{cases}
\]

Note that \(Hf\) is defined on the circle \(\mathbb{T}\). From the first set of formulas we also see that \(H\) commutes with differentiation \(\partial\tau\) with respect to the polar angle \(\tau\)

\[
\partial\tau Hu = H\partial\tau u.
\]

**Theorem 10.** If \(1 < p < \infty\) the Hilbert operator \(H : L^p(\mathbb{T}) \to L^p(\mathbb{T})\) is bounded.

Remark. The Hilbert operator is not continuous in \(L^1\) and \(L^\infty\). Also there are continuous functions \(u\) for which \(Hu\) is unbounded.

The most important property of the Hilbert operator is that it connects real and imaginary part of holomorphic functions on \(\mathbb{T}\).

**Theorem 11.** If \(w = u + iv \in L^p(\mathbb{T})\) with \(1 < p < \infty\), then \(w \in H^p(\mathbb{T})\) if and only if the boundary functions of \(u\) and \(v\) are connected by the relations

\[
v = -Hu + v(0), \quad u = Hv + u(0).
\]
Since the real part of a holomorphic function is a harmonic conjugate of its imaginary part, $H$ is also called the conjugation operator.

For convenient reference the mapping properties of all operators in this chapter are summarized in the table below. Here $t = e^{it}$ stands for a variable on $\mathbb{T}$, while $z \in \mathbb{D}$.

<table>
<thead>
<tr>
<th>function $t^k, k &gt; 0$</th>
<th>Cauchy $z^k$</th>
<th>Poisson $z^k$</th>
<th>Schwarz $2z^k$</th>
<th>Riesz $tk$</th>
<th>Hilbert $ikt^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$t^k, k &lt; 0$</td>
<td>$0$</td>
<td>$z^{-k}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-ikt^k$</td>
</tr>
</tbody>
</table>

3 Linear Riemann-Hilbert Problems

As we just saw in the preceding chapter, holomorphic functions in $H^p$ are uniquely determined by their boundary values. However, the boundary function cannot be prescribed arbitrarily, since it has to have vanishing Fourier coefficients with negative indices. In other words, the Dirichlet problem for holomorphic functions is not well posed.

This observation leads to the question of correct (‘natural’) boundary value problems for holomorphic functions; a problem that was already studied by Bernhard Riemann in 1851.

When looking for well-posed boundary value problems for holomorphic functions, it is helpful to keep in mind that the Laplace operator factors as a product of the two Cauchy-Riemann-type operators $\partial/\partial \overline{z}$ and $\partial/\partial z$. Since the Dirichlet problem for the Laplace equation is well posed, this leads to the idea of considering boundary value problems for holomorphic functions, where ‘half of the boundary function’ is given. The simplest problem of this kind was already studied in the preceding section: Find a holomorphic function $w$ with prescribed real part on the boundary

$$\text{Re } w = u \quad \text{on } \mathbb{T}.$$  

In fact, the solution of this problem is essentially unique (Schwarz’ integral formula). In this section we investigate more general linear boundary value problems of this type, so called Riemann-Hilbert problems.

Throughout this chapter we assume that $p$ is a fixed number with $1 < p < \infty$. We formulate the results for arbitrary $p$, the proofs are usually given for $p = 2$ only.

The notation $w = u + iv$ is exclusively used for a holomorphic function with real part $u$ and imaginary part $v$.

The linear Riemann-Hilbert problem consists in finding all holomorphic functions $w = u + iv \in H^p$, such that the real and the imaginary part of the boundary function satisfy a (real) linear relation

$$a(t)u(t) + b(t)v(t) = c(t)$$

(1)
almost everywhere on $T$. Here $a, b,$ and $c$ are given real–valued functions on $T$.

Another frequently used form of writing the boundary condition is

$$\text{Im}(f(t)w(t)) = c(t),$$  \hspace{1cm} (2)$$

with a given complex-valued function $f = b - ia$, called the symbol of the problem (recall that $\overline{f(t)}$ denotes the complex conjugate of $f(t)$). Geometrically the boundary condition requires that every boundary value $w(t)$ must lie on a straight straight line which is parallel to $f(t)$, considered as a vector in $\mathbb{R}^2$.

We study problem (2) for continuous symbols $f$ which have no zeros on $T$. Later on the results are extended to piecewise continuous symbols. The right-hand side $c$ is assumed to be in the Lebesgue space $L^p_T$ of real–valued functions, the solution is sought in the Hardy space $H^p$.

It turns out that the solvability of (2) depends on a geometric characteristic of its symbol $f$. If $f$ is continuous and nonvanishing on $T$ we can choose a branch of the argument $\varphi := \text{arg } f$ such that the mapping

$$\Phi: [0, 2\pi] \to \mathbb{R}, \quad \tau \mapsto \text{arg } f(e^{i\tau})$$

is continuous. The winding number of $f$ (about the origin) is then defined as

$$\text{wind } f := \frac{1}{2\pi} (\Phi(2\pi) - \Phi(0)).$$

Note that wind $f$ is an integer and does not depend on the special choice of the branch of the argument. Geometrically, the winding number of $f$ indicates the number of oriented turns of the graph of $f$ about the origin. The number wind $f$ is also referred to as the (geometrical) index of the Riemann-Hilbert problem (1).

### 3.1 Problems with index zero

After dividing the boundary condition $\text{Re}(\overline{f}w) = c$ by $|f|$, we may assume that $|f| \equiv 1$ on $T$. If the function $f$ has winding number zero it can be written as $f = \exp(i\varphi)$ with a continuous function $\varphi$. We start with investigating the homogeneous Riemann-Hilbert problem.

**Theorem 12.** Let $f = \exp(i\varphi)$, $\varphi \in C(T)$, and $1 < p < \infty$. Then the general solution $w \in H^p$ of the homogeneous Riemann-Hilbert problem

$$\text{Im}(\overline{f}w) = 0$$  \hspace{1cm} (3)$$

is given by $w = d \cdot w_0$, where $d$ is an arbitrary real number and

$$w_0 := \exp(H\varphi + i\varphi).$$
Proof. Before we give a rigorous proof of the result, we derive a formal solution.

1. The function \( w_0 := \exp(H\varphi + i\varphi) \) is holomorphic and satisfies
\[
\arg w_0 = \varphi = \arg f.
\]
Since \( \arg(\overline{f}w_0) = -\arg f + \arg w_0 = 0 \), the function \( w_0 \) is a (special) solution of the homogeneous Riemann-Hilbert problem (3).

2. Conversely, if \( w \) is any solution of the homogeneous problem, we consider the function \( w/w_0 \). Since \( \Im (w/w_0) = 0 \) on \( \mathbb{T} \), we have \( w/w_0 = C \) with a real constant \( C \). Thus the general solution is \( w = Cw_0 \).

3. In order to justify these formal manipulations we need Zygmund’s lemma. Roughly speaking this estimate shows that \( H\varphi \) has only logarithmic singularities (of a definite strength) if \( \varphi \) is bounded.

Lemma 1. (Zygmund) For all \( \varphi \in L^\infty(\mathbb{T}) \) with \( \|\varphi\|_\infty \leq \gamma < \pi/2 \) we have
\[
\| \exp(H\varphi) \|_1 \leq \frac{1}{\cos \gamma}.
\]
If \( \varphi \in C(\mathbb{T}) \), then \( \exp(H\varphi) \in L^p(\mathbb{T}) \) for all \( p \) with \( 1 \leq p < \infty \).

Proof. We define the holomorphic function \( f \) as the Schwarz’ integral of \( \varphi \),
\[
f(z) := \frac{1}{2\pi} \int_{\mathbb{T}} \frac{t+z}{t-z} \varphi(t) |dt|.
\]
Since \( f \in H^p \) for all \( p < \infty \), it has angular limits almost everywhere on \( \mathbb{T} \) and the boundary function of \( f \) equals \( \varphi - iH\varphi \). The function \( g := \exp(if) \) is holomorphic in \( \mathbb{D} \) and has angular limits almost everywhere on \( \mathbb{T} \). Its boundary function satisfies \( g^* = \exp(if^*) = \exp(H\varphi + i\varphi) \), i.e. \( |g| = \exp(H\varphi) \) almost everywhere on \( \mathbb{T} \). Since \( \Im f(0) = 0 \),
\[
\Re g(0) \leq |g(0)| = |\exp(if(0))| = 1.
\]
Moreover,
\[
|\arg g(z)| = |\Re f(z)| \leq \|\varphi\|_\infty \leq \gamma < \pi/2.
\]
So, for each \( r < 1 \),
\[
1 \geq \Re g(0) = \frac{1}{2\pi} \int_{\mathbb{T}} \Re g(rt) |dt| = \frac{1}{2\pi} \int_{\mathbb{T}} |g(rt)| \cos \arg g(rt) |dt| \geq \frac{\cos \gamma}{2\pi} \int_{\mathbb{T}} |g(rt)| |dt|.
\]
Consequently \( g \in H^1 \) and \( \|g\|_1 \leq 1/\cos \gamma \). Since \( \exp(H\varphi) = |g| \) on \( \mathbb{T} \), assertion (i) follows.
In order to prove the second assertion we write \( \varphi \) as a sum of a trigonometric polynomial and a function with a sufficiently small \( L^\infty \)-norm. \( \square \)
We use Zygmund’s lemma to finish the proof of Theorem 12. First of all we see that the (boundary) function
\[ w_0 = \exp(H\varphi + i\varphi) = \exp(H\varphi) \cdot \exp(i\varphi) \]
belongs to \( L^p \) for all \( p \in [1, \infty) \). The function \( H\varphi + i\varphi \) extends holomorphically into \( \mathbb{D} \) and belongs \( H^p \). Consequently \( w_0 \), the exponential of that function, is holomorphic in \( \mathbb{D} \). Its nontangential limits exist almost everywhere and the boundary function is in \( L^p \), consequently we have \( w_0 \in H^p \). It was already shown that it is a (special) solution of the homogeneous Riemann-Hilbert problem.

By the same reasoning, \( 1/w_0 \) belongs to \( H^q \) for each \( q \in [1, \infty) \). If \( w \in H^p \) is any solution, then (by Hölder’s inequality) \( w/w_0 \in H^{1+\varepsilon} \) for some positive \( \varepsilon \). Since \( \text{Im} (w/w_0) = 0 \) on \( \mathbb{T} \), the function \( w/w_0 \) must be a real constant. \( \square \)

In the next step we consider inhomogeneous problems with index zero.

**Theorem 13.** Let \( f = \exp(i\varphi) \), \( \varphi \in C(\mathbb{T}) \), and \( 1 < p < \infty \). Then, for each real valued function \( c \in L^p(\mathbb{T}) \), the general solution of \( \text{Im} (\overline{f} \ w) = c \) in \( H^p \) is given by
\[ w = w_c + d \cdot w_0, \]
where \( d \) is an arbitrary real number and
\[ w_0 := \exp(H\varphi + i\varphi), \]
\[ w_c := w_0 \cdot \left( H \left( c/|w_0| \right) + i \left( c/|w_0| \right) \right). \]

The operator \( L^p \to H^p \), \( c \mapsto w_c \) is bounded.

**Proof.** (For \( p = 2 \)) We start with a formal construction of the solution. What follows resembles the method of variation of constants from ordinary differential equations.

1. We seek a particular solution of the inhomogeneous problem (2) in the form
\[ w(t) = C(t) \ w_0(t) \]
with a holomorphic function \( C \). Inserting this ansatz into the boundary relation gives
\[ \text{Im} C(t) = \frac{c(t)}{|w_0(t)|} \]
and, reconstructing the holomorphic function \( C \) from its imaginary part, we get the desired formula for the solution.

2. It remains to justify this formal approach. We have to show that the special solution \( w_c \) of the inhomogeneous problem belongs to \( H^2 \) and that \( \|w_c\|_2 \leq C \|c\|_2 \) for some \( C > 0 \). Since
\[ |w_c| = |i \ c + |w_0| H(c/|w_0|)| \leq |c| + |w_0| |H(c/|w_0|)|, \]
we have
\[ \|w_c\|_2 \leq \|c\|_2 + \|w_0\|_2 \|H(c/|w_0|)| \leq C \|c\|_2 \|w_0\|_2, \]
with \( C = \|H\|_2 \). The proof is complete. \( \square \)
and \(|w_0| = \exp(H\varphi)|
 we see that it suffices to show that
\[ \|\exp(H\varphi) \cdot H(c \exp(-H\varphi))\|_2 \leq C \|c\|_2.\]

This estimate follows from a result of Helson and Szegö about the mapping properties of the Hilbert operator in weighted Lebesgue spaces
\[ L^2_\varphi(T) := \{ f : \|f\|_{2,\varphi} := \|\varphi f\|_2 < \infty \} \]
on \(T\).

**Theorem 14.** (Helson and Szegö) Let \(\varphi\) be an almost everywhere positive weight function on \(T\). Then the Hilbert operator \(H\) is bounded in \(L^2_\varphi(T)\) if and only if \(\varphi\) admits a representation
\[ \varphi = \exp(\psi + H\varphi), \]
where \(\psi, \varphi \in L^\infty(T)\) and \(\|\varphi\|_\infty \leq \gamma < \pi/4\). If \(\psi = 0\), then
\[ \|\varphi H v\|_2 \leq C(\gamma) \|\varphi v\|_2 \]
with \(C(\gamma) := 1/(\sqrt{2} \cos \gamma - 1)\).

**Proof.** We only prove the ‘if’ part since we do not need the converse. The function \(\exp \psi\) is bounded from above and below (away from zero), and thus we can restrict attention to weight functions \(\varphi := \exp(H\varphi)\) with \(\|\varphi\|_\infty \leq \gamma < \pi/4\). Moreover, taking into account that \(H\) annihilates the constants and that \(H\varphi(0) = 0\), it is sufficient to show that there exists a constant \(C(\gamma)\), such that for all polynomials \(w = u + iv\) with \(w(0) = 0\) the estimate
\[ \|\varphi u\|_2 \leq C(\gamma) \|\varphi v\|_2 \]
holds. The result then follows from a density argument. By Zygmund’s lemma, the function \(\exp(2H\varphi)\) is in \(L^1\), and hence \(g := \exp(H\varphi + i\varphi)\) satisfies
\[ g^2 = \cos 2\varphi \exp(2H\varphi) + i \sin 2\varphi \exp(2H\varphi) \in H^1. \]

Since the function \(w\) is a polynomial, also \(w^2 g^2 \in H^1\). Because \(w(0) = 0\),
\[ \frac{1}{2\pi} \int_T w^2(t) g^2(t) |dt| = (w^2 g^2)(0) = w^2(0)g^2(0) = 0. \]

From \(w^2 = u^2 - v^2 + 2i uv\) we obtain that
\[ \text{Re} (w^2 g^2) = g^2 (u^2 - v^2) \cos 2\varphi - 2 g^2 uv \sin 2\varphi, \]
which implies
\[ \int_T g^2 (v^2 - u^2) \cos 2\varphi |dt| + \int_T 2 g^2 uv \sin 2\varphi |dt| = 0. \]
Now we have \( \cos 2\varphi \geq \delta := \cos 2\gamma > 0 \) and thus
\[
\delta \| \varrho u \|_2^2 \leq \| \varrho v \|_2^2 + 2 \| \varrho u \|_2 \| \varrho v \|_2,
\]
which finally gives the desired result
\[
\| \varrho u \|_2 \leq \frac{1}{\delta} (1 + \sqrt{1 + \delta}) \| \varrho v \|_2 = C(\gamma) \| \varrho v \|_2.
\]

We continue with the proof of Theorem 13. In order to estimate the norm of \( w_c \) let \( \varrho := \exp(H\varphi) \). It is sufficient to show that
\[
\| H(\varrho^{-1}c) \|_{2,\varrho} \equiv \| \varrho H(\varrho^{-1}c) \|_2 \leq C \| c \|_2 \equiv C \| \varrho^{-1}c \|_{2,\varrho},
\]
which is equivalent to the boundedness of \( H \) in \( L^2_\varrho \). According to Theorem 14 this estimate holds if \( H\varphi \) admits a representation \( \psi + H\varphi_1 \), where \( \psi, \varphi_1 \in L^\infty \) and \( \| \varphi_1 \| \leq \pi/4 \). In order to get such a representation we write the (continuous) function \( \varphi \) as the sum of a trigonometric polynomial and a function \( \varphi_1 \) with sup–nom less than \( \pi/4 \).

Frequently the Riemann-Hilbert problem is studied together with an additional condition in order to fix the value of the free constant \( d \). Since the solution is not necessarily continuous on \( \mathbb{D} \), it makes no sense to impose a condition for values of the solution at a point on \( \mathbb{T} \). A typical side condition involves the value \( w(0) \) of the solution at \( z = 0 \),

\[
\alpha u(0) + \beta v(0) = \gamma. \tag{4}
\]

A straightforward computation yields the value of the homogeneous solution \( w_0 \) at \( z = 0 \),
\[
w_0(0) = \exp(i\varphi(0)) = \cos\delta + i \sin\delta,
\]
where
\[
\delta := \frac{1}{2\pi} \int_0^{2\pi} \arg f(e^{i\tau}) \, d\tau.
\]

Consequently, the Riemann-Hilbert problem \( \text{Im} (\mathcal{F} w) = c \) with the side condition (4) is uniquely solvable if \( \alpha \cos\delta + \beta \sin\delta \neq 0 \).

### 3.2 Problems with positive index

In this and the following section we investigate Riemann-Hilbert problems with continuous coefficients and arbitrary index \( \kappa := \text{wind } f \). Again we can restrict ourselves to symbols with \( |f| \equiv 1 \).

**Theorem 15.** Let \( 1 < p < \infty \), \( f \in C(\mathbb{T}) \), \( |f| \equiv 1 \) on \( \mathbb{T} \), and \( \kappa := \text{wind } f > 0 \). Then the homogeneous Riemann-Hilbert problem \( \text{Im} (\mathcal{F} w) = 0 \) has \( 2\kappa + 1 \) (real) linearly independent solutions and the inhomogeneous problem \( \text{Im} (\mathcal{F} w) = c \) is solvable in \( H^p \) for any right-hand side \( c \in L^p_B \).
Proof. Let \( g(z) := C_0 + C_1 z + \ldots + C_{\kappa-1} z^{\kappa-1} \) be any polynomial of degree \( \kappa - 1 \). If \( \tilde{w} \) is a solution of the Riemann-Hilbert problem

\[
\text{Im} \left( \overline{ft^{-\kappa}} \tilde{w} \right) = c - \text{Im} \left( \overline{f} g \right)
\]

with index zero, then \( w \) defined by \( w(z) := z^{\kappa} \tilde{w}(z) + g(z) \) is a solution of \( \text{Im} \left( \overline{f} w \right) = c \). For \( c \equiv 0 \) the transformed problem has a one dimensional solution space \( dw_0 \). All solutions obtained by varying \( C_0, \ldots, C_{\kappa} \) are linearly independent.

Conversely, any solution \( w \) of the original problem is obtained this way by choosing \( g \) such that \( w - g \) has a zero of order \( \kappa - 1 \) at \( z = 0 \).

It follows from the proof that the solution can be made unique by imposing side conditions

\[
\text{Im} (\overline{\delta} w(0)) = \gamma, \quad w(z_j) = w_j \quad (j = 1, \ldots, \kappa).
\]

at 0 and pairwise distinct points \( z_j \in \mathbb{D} \setminus \{0\} \). While \( w_j \in \mathbb{C} \) and \( \gamma \in \mathbb{R} \) can be chosen arbitrarily, \( \delta \) must satisfy a compatibility condition which depends on \( f \) and the choice of the points \( z_j \).

### 3.3 Problems with negative index

Finally we consider the linear Riemann-Hilbert problem with negative index.

**Theorem 16.** Let \( 1 < p < \infty \), \( f \in C(\mathbb{T}) \), \( |f| \equiv 1 \) on \( \mathbb{T} \), and \( \kappa := \text{wind} f < 0 \). Then the homogeneous Riemann-Hilbert problem

\[
\text{Im} \left( \overline{f} w \right) = 0
\]

has only the trivial solution \( w \equiv 0 \) in \( H^p \). The inhomogeneous problem

\[
\text{Im} \left( \overline{f} w \right) = c
\]

is solvable if and only if the 2\(|\kappa| - 1 \) solvability conditions

\[
\int_0^{2\pi} c(e^{i\tau}) \psi(e^{i\tau}) \cos k\tau \, d\tau = 0, \quad k = 0, 1, \ldots, |\kappa| - 1,
\]

\[
\int_0^{2\pi} c(e^{i\tau}) \psi(e^{i\tau}) \sin k\tau \, d\tau = 0, \quad k = 1, \ldots, |\kappa| - 1,
\]

are satisfied, where

\[
\psi := \exp \left( -H \arg(t^{\kappa}|f|) \right).
\]

**Proof.** With \( \tilde{w}(z) := z^{-\kappa} w(z) \) and \( g(t) := t^{-\kappa} f(t) \) we have

\[
\text{Im} \left( \overline{f(t)} w(t) \right) = \text{Im} \left( \overline{f(t)} t^{-\kappa} t^{-\kappa} w(t) \right) = \text{Im} \left( \overline{g(t)} \tilde{w}(t) \right).
\]

The function \( w \) is in \( H^p \) and solves the Riemann-Hilbert problem \( \text{Im} \left( \overline{f} w \right) = c \) if and only if \( \tilde{w} \) is in \( H^p \), solves \( \text{Im} \left( \overline{g} \tilde{w} \right) = c \), and has a zero of order \( |\kappa| \) at \( z = 0 \).
Since wind \( g = 0 \), we have an explicit expression for \( \tilde{w} \). The solvability conditions are obtained by expressing
\[
\tilde{w}(0) = 0, \quad \frac{d\tilde{w}}{dz}(0) = 0, \quad \ldots, \quad \frac{d^{|\kappa|-1}\tilde{w}}{dz^{|\kappa|-1}}(0) = 0
\]
in terms of the Fourier coefficients of the boundary functions. Note that the number of conditions is reduced by one, since \( \tilde{w} \) involves a free constant which can be used to eliminate one condition.

Note that the solvability conditions are given by bounded linear functionals in \( L^p \) which act on the right–hand side \( c \).

### 3.4 Problems with piecewise continuous coefficients

In the final section of this chapter we suppose that \( f \) is continuous with the possible exception of finitely many jumps at points \( t_1, t_2, \ldots, t_n \in \mathbb{T} \). We assume that the points \( t_j = e^{i\tau_j} \) are arranged in counterclockwise order, such that \( 0 \leq \tau_1 < \tau_2 < \ldots < \tau_n < 2\pi \), and denote the one–sided limits of \( f(e^{i\tau}) \) at \( \tau_j \) by \( f_-(t_j) \) and \( f_+(t_j) \). As always we normalize the boundary condition such that \(|f| \equiv 1\).

**Lemma 2.** Let \( f : \mathbb{T} \to \mathbb{T} \) be piecewise continuous. Then \( f \) has a representation
\[
f(t) = s(t) \cdot t^\kappa \cdot f_0(t) \cdot f_r(t)
\]
where \( s : \mathbb{T} \to \{+1, -1\} \) is piecewise continuous, \( f_0 : \mathbb{T} \to \mathbb{T} \) is a Laurent polynomial (i.e. a finite linear combination of integer powers of \( t \)) with wind \( f_0 = 0 \), \( f_r(t) : \mathbb{T} \to \mathbb{T} \) is piecewise continuous with \( |\arg f_r| \leq \pi/4 \), and \( \kappa \in \mathbb{Z}/2 \). If \( \text{Re} \left( f_+(t_j)/f_-(t_j) \right) \neq 0 \), then \( \sup |\arg f_r| < \pi/4 \) and the number \( \kappa \) is uniquely defined.

Note that \( t^\kappa \) stands for any branch of this function which is continuous on \( \mathbb{T} \setminus \{1\} \). The proof is constructive and elementary and we leave it as an exercise.

In order to formulate the main result in the language of functional analysis, we recall that a bounded linear operator \( A \) is said to be *Fredholm* if it has closed range and if the dimension of its kernel and the codimension of its image are finite. In this case
\[
\text{ind } A := \dim \text{ker } A - \text{codim im } A
\]
is called the (functional analytic) *index* of the operator \( A \).

**Theorem 17.** Assume that \( f : \mathbb{T} \to \mathbb{T} \) is piecewise continuous with jumps at \( t_1, t_2, \ldots, t_n \). Then the Riemann–Hilbert operator
\[
R_f : H^2(\mathbb{T}) \to L^2_\mathbb{R}(\mathbb{T}), \quad w \mapsto \text{Im } (\overline{f} w)
\]
has the following properties.
(i) If $\text{Re } f_+ (t_j)/f_- (t_j) \neq 0$ for all $j = 1, \ldots, n$, then $R_f$ is Fredholm with

$$\dim \ker R_f = \max \{0, 2\kappa + 1\}, \quad \text{codim } \text{im } R_f = \max \{0, -2\kappa - 1\}.$$ 

In particular $R_f$ is invertible if and only if $\kappa = -1/2$.

(ii) If $\text{Re } f_+ (t_j)/f_- (t_j) = 0$ for some $j$, then $R_f$ is not Fredholm.

Proof. 1. Let $f(t) = s(t) \cdot t^{\kappa} \cdot f_0(t) \cdot f_r(t)$ be the representation of the symbol $f$ according to Lemma 2. Multiplying the boundary condition by $s$ does not change much, only $c$ may be replaced by $-c$ in some arcs. So we get an equivalent problem with $s \equiv 1$. Moreover, we have $\kappa = \mu$ or $\kappa = \mu - 1/2$ with $\mu \in \mathbb{Z}$, and the representation $f(t) = t^{\mu} g(t)$, together with the techniques already used for continuous symbols, transform the problem to one with either $\kappa = 0$ or $\kappa = -1/2$.

2. Let the assumptions of (i) be satisfied with $\kappa = 0$. Then we have $f = \exp i(\varphi_0 + \varphi_r)$, where $\varphi_0 := \arg f_0$ is a trigonometric polynomial (in $\tau$), and $\varphi_r := \arg f_r$ has sup–norm less than $\pi/4$. The rest of the proof is exactly the same like for continuous symbols, with the only difference that now the Helson-Szegő theorem is applied with $\varphi = \varphi_r$ and $\psi = H \varphi_0$.

3. If the assumptions of (i) are satisfied with $\kappa = -1/2$ we have

$$f(t) = t^{-1/2} \cdot f_0(t) \cdot f_r(t),$$

with a continuous branch of $t^{-1/2}$ on $\mathbb{T} \setminus \{1\}$. Then the functions $g$ and $d$ defined by

$$g(t) := \begin{cases} f(t^2) & \text{if } 0 \leq \arg t < \pi \\ -f(t^2) & \text{if } \pi \leq \arg t < 2\pi, \end{cases} \quad d(t) := \begin{cases} c(t^2) & \text{if } 0 \leq \arg t < \pi \\ -c(t^2) & \text{if } \pi \leq \arg t < 2\pi, \end{cases}$$

are odd and $g$ has (generalized) winding number $-1$. If $w$ is a solution of $\text{Im } (\mathcal{J} w) = c$, then $\hat{w}$ defined by $\hat{w}(z) := w(z^2)$ is a solution of $\text{Im } (\mathcal{G} \hat{w}) = d$. Conversely, if $\hat{w}$ solves the second problem, then $\hat{w}$ must be even. Indeed, otherwise $\hat{w}_-$ defined by $\hat{w}_-(z) = \hat{w}(-z)$ would be another solution of this problem, which is impossible since wind $g = -1$. This ensures that the relation $w(z^2) = \hat{w}(z)$ defines a function $w \in H^2$. It remains to show that $\text{Im } (\mathcal{G} \hat{w}) = d$ always has a solution. This happens if and only if $\text{Im } (\mathcal{G} \hat{w}) = d$ always has a solution $\hat{w}$ with $\hat{w}(0) = 0$. The problem has index zero, so a solution $\hat{w}$ exists. Since $t g$ is even and $d$ is odd, the function $\hat{w}_-$ defined by $\hat{w}_-(z) := -\hat{w}(-z)$ is also a solution and thus $(\hat{w} + \hat{w}_-)/2$ is a solution that vanishes at $z = 0$.

4. It remains to prove (ii). Assume to the contrary that $R_f$ is Fredholm. Then there are arbitrarily small (in the sup-norm) perturbations of $f$ which satisfy (i) and have different winding numbers. This is in conflict with the stability of the Fredholm index with respect to small perturbations of $R_f$. 

\[\square\]
4 Nonlinear Riemann-Hilbert problems

Surprisingly, the history of boundary value problems for holomorphic functions does not start with linear problems. Careful reading of Riemann’s thesis reveals the following problem, which is now frequently also referred to as the nonlinear Riemann-Hilbert problem:

Let \( \{M_t\}_{t \in \mathbb{T}} \) be a given family of curves in the complex plane. Find all functions \( w = u + iv \) holomorphic in the open unit disk \( \mathbb{D} \) and continuous in the closed unit disk \( \overline{\mathbb{D}} \) (this space of functions is called the disc algebra and we denote it by \( H^\infty \cap C \)) such that the boundary condition

\[ w(t) \in M_t \]

is satisfied for all points \( t \in \mathbb{T} \). The curves \( M_t \) go by the name target curves or restriction curves of the problem. The figure illustrates this definition.

For linear Riemann-Hilbert problems \( au + bv = c \) the target curves are the straight lines

\[ M_t := \{ u + iv \in \mathbb{C} : a(t)u + b(t)v = c(t) \} \].

A less geometrical form of writing the boundary condition of a nonlinear Riemann-Hilbert problem is

\[ F(t, u(t), v(t)) = 0. \]

Here \( F: \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R} \) is a given function, called the defining function of the problem. Note that the defining function is by no means unique, only the zero set of \( F \) is important to define the target curves \( M_t \).

4.1 Explicit Riemann-Hilbert problems

This section is devoted to the case where the defining function can be solved with respect to one variable, \( v \) say, which results in boundary conditions of the type

\[ v(t) = F(t, u(t)) \tag{5} \]

Let us look for the right spaces to treat the problem. Since \( F: \mathbb{T} \times \mathbb{R} \to \mathbb{R} \) will be differentiated (for technical reasons) we assume it to be continuously differentiable.
Further we need the Hilbert operator, which acts continuously in Lebesgue spaces $L^p(\mathbb{T})$, but not in $C(\mathbb{T})$ and, more generally, in $C^k(\mathbb{T})$. The right–hand side of (5) contains the superposition operator $u \mapsto F(., u)$. Since this operator is (in general) not well defined in Lebesgue spaces, we try next Sobolev spaces on the unit circle. Recall that the Sobolev space $W^1_p(\mathbb{T})$ is the linear space of all functions $f \in L^p(\mathbb{T})$ which have generalized derivatives $\partial_\tau f \in L^p(\mathbb{T})$ with respect to the polar angle $\tau$. The norm in $W^1_p(\mathbb{T})$ is given by
\[
\|f\|_{W^1_p} = \left\{ \|f\|_p^p + \|\partial_\tau f\|_p^p \right\}^{1/p}.
\]
The functions in $W^1_p(\mathbb{T})$ are continuous and the embedding $W^1_p(\mathbb{T}) \hookrightarrow C(\mathbb{T})$ is compact. Then, in fact, all involved operators are bounded and continuous in $W^1_p$.

The main result of this chapter is the existence of a unique solution under relatively mild assumptions on the defining function $F$.

**Theorem 18.** Assume that the function $F: \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable and has bounded derivatives. Then, for each $u_0 \in \mathbb{R}$, the nonlinear Riemann-Hilbert problem
\[
v(t) = F(t, u(t)), \quad u(0) = u_0
\]
has a unique solution $w = u + iv$ in $H^\infty \cap C$. This solution belongs to $H^\infty \cap W^1_p$ for all $p$ with $1 \leq p < \infty$.

The proof is based on Schauder’s fixed point theorem (see [2], Sect. 7.1 [3], Sect. 2.8). Recall that a (nonlinear) operator $K: E \to E$ is said to be compact, if it is continuous and the image of each bounded subset is relatively compact. The following result is the simplest version of Schauder’s fixed point theorem, but sufficient for our purposes.

**Theorem 19.** (Schauder) If the operator $K: E \to E$ is compact and has a bounded range $K(E)$, then $K$ has a fixed point.

In the following we transform the nonlinear Riemann-Hilbert problem to a fixed point equation for a compact operator $K$ in the Sobolev space $E := W^1_p(\mathbb{T})$. Note that we can rewrite the problem as a nonlinear operator equation $v = F(., -Hv + u_0)$, which already has fixed point form. However, the Hilbert operator $H$ is not compact, and so we need a trick to produce compactness.

The idea consists in differentiating the boundary relation $v(e^{i\tau}) = F(e^{i\tau}, u(e^{i\tau}))$ with respect to the polar angle $\tau$. This results in a quasilinear Riemann Hilbert problem for the derivative $\partial_\tau w$ and integrating its solution along $\mathbb{T}$ reproduces $u$. Indeed this works fine, but the details are a bit more involved.

For $u \in W^1_p$ we define
\[
a := \partial_u F(., u - u(0) + u_0), \quad c := \partial_\tau F(., u - u(0) + u_0)
\]
and denote by $\tilde{w} \equiv \tilde{u} + i \tilde{v}$ the solution to the linear Riemann-Hilbert problem
\[
\tilde{v} - a \tilde{u} = c, \quad \tilde{u}(0) = 0.
\]
Further we set
\[ \hat{u}(e^{i\tau}) := \int_0^\tau \tilde{u}(e^{i\sigma}) \, d\sigma. \] (9)

This definition is correct since the mean value of \( \tilde{u} \) vanishes due to the condition \( \tilde{u}(0) = 0 \) imposed on \( \tilde{u} \). The equations (7)–(9) define the (nonlinear) operator
\[ K : W^1_p \to W^1_p, \quad u \mapsto \hat{u} - \hat{u}(0) + u_0. \]

Two relevant properties of \( K \) are summarized in the next lemma.

**Lemma 3.** Let \( 1 < p, q < \infty \). Then the operator \( K \) has the following properties.

(i) The mapping \( K : W^1_p \to W^1_q \) is compact.

(ii) A function \( u \in W^1_p \) is a fixed point of \( K \) if and only if there exists \( d \in \mathbb{R} \) such that
\[ Hu + F(., u) = d \quad \text{and} \quad u(0) = u_0. \] (10)

Condition (10) is the same as saying that \( w = u + i(d - Hu) \) is a solution of the Riemann-Hilbert problem (6).

**Proof.**

1. The operator \( K \) is composed as shown in the following scheme:
\[
W^1_p \to C \to C \times C \to L^q \to W^1_q
\]
\[
u \mapsto u \mapsto (a, c) \mapsto \hat{u} \mapsto \tilde{u}.
\]
All mappings involved in this diagram are continuous and the first one is a compact embedding. This proves (i).

2. Let \( w \equiv u + iv \) satisfy \( Hu + F(., u) = \text{const} \) and \( u(0) = u_0 \). Differentiation with respect to the polar angle \( \tau \) yields
\[
H \partial_\tau u + \partial_\tau F(., u) + \partial_a F(., u) \cdot \partial_\tau u = 0,
\]
\[
H \partial_\tau u + c + a \cdot \partial_\tau u = 0.
\]
Since the function \( \partial_\tau u - iH \partial_\tau u \) is holomorphic and \( (\partial_\tau u)(0) = 0 \), it must coincide with the unique solution \( \tilde{w} \) of (8), and hence \( \partial_\tau u = \tilde{u}, \ H \partial_\tau u = \tilde{v} \). Consequently,
\[
K u := \hat{u} - \tilde{u}(0) + u_0 = \int_0^\tau \partial_\tau u(e^{i\sigma}) \, d\sigma + \text{const} = u + \text{const}.
\]
Since \( K u(0) = u_0 = u(0) \), the constant vanishes.

3. If, conversely, \( u \) is a fixed point of \( K \), then \( \tilde{u} := \hat{u} - \hat{u}(0) + u_0 = u \) and thus \( u(0) = u_0 \). Further,
\[
\tilde{u} = \partial_\tau \hat{u} = \partial_\tau u,
\]
and therefore
\[
\tilde{v} = -H\tilde{u} + \tilde{v}(0) = -H \partial_\tau u + \tilde{v}(0),
\]
which implies
\[ \partial_\tau (F(., u) + Hu) = \partial_u F(., u) \cdot \partial_\tau u + H \partial_\tau u + \partial_\tau F(., u) = a \tilde{u} - \tilde{v} + \tilde{v}(0) + c = \text{const}. \]

The mean value of the function on the left-hand side is zero, and hence the constant on the right-hand side must vanish also, so that
\[ F(., u) + Hu = \text{const}. \]

This completes the proof of Lemma 3.

Once the existence of a fixed point \( u \) of \( K \) is established, the solutions \( w \) of the Riemann-Hilbert problem is simply given by \( w := u + i F(., u) \).

In order to apply Schauder’s fixed point theorem, we prove that \( K \) maps \( W^1_p \) into a certain ball, provided that \( p \) is sufficiently close to 1.

**Lemma 4.** There exists a \( p > 1 \) such that the range of \( K : W^1_p \rightarrow W^1_p \) is bounded.

The proof of this result requires some preparations. Above all we need a better estimate of the solutions to the linear Riemann-Hilbert problem
\[ \tilde{v} - a \tilde{u} = c, \quad \tilde{u}(0) = 0. \]

Recall that its solutions are given by the following set of formulas:
\[ c_0 := c/|1 + ia|, \quad \varphi := \arg (1 + ia), \quad w_0 := \exp (H \varphi + i \varphi) \]
\[ \tilde{w} = Bc := w_0 \cdot \left( H \left( c_0/|w_0| \right) + i c_0/|w_0| \right). \]

We already know that \( B : L^p \rightarrow L^p \) is bounded, but we get a better estimate for the norm of \( B \) if we think of \( B \) as of a mapping \( B : L^q \rightarrow L^p \) with \( p \) close to 1 and \( q \) sufficiently large. To this end we adapt the Helson-Szegö theorem to pairs of weighted Lebesgue spaces. With respect to the situation of interest only weights \( \varrho = \exp (H \varphi) \) are considered.

**Lemma 5.** For each \( \gamma \) with \( 0 \leq \gamma < \pi/2 \) there exist positive numbers \( C, p, \) and \( q \) with \( 1 < p < 2 < q < \infty \), such that for all weight functions \( \varrho \) with
\[ \varrho := \exp (H \varphi), \quad ||\varphi||_\infty \leq \gamma < \pi/2, \]
the norm of \( H : L^q_\varrho \rightarrow L^p_\varrho \) is bounded by \( C \),
\[ ||\varrho Hu||_p \leq C ||\varrho u||_q. \]
Proof. We choose an $\varepsilon$ with $0 < \varepsilon < 1$ such that
\[
\delta := \frac{1 + \varepsilon}{1 - \varepsilon} \gamma < \frac{\pi}{2}
\]
and prove that the assertion of the lemma is valid for $p := 1 + \varepsilon$, $q := 2 + 2/\varepsilon$, and a certain positive constant $C$. By Hölder’s inequality,
\[
\|\varrho H u\|_{p}^{p} = \left( \int \varrho^{p} |H u|^{p} \, d\tau \right)^{\frac{1}{p}} \left( \int \varrho^{\frac{2}{1-\varepsilon}} \left( \varrho |H u|^{2} \right)^{\frac{p}{2}} \, d\tau \right)^{\frac{1}{2}} \leq \left\{ \int \varrho^{\frac{2}{1-\varepsilon}} \, d\tau \right\}^{1 - \frac{q}{p}} \cdot \left\{ \int \varrho |H u|^{2} \, d\tau \right\}^{\frac{q}{2}}.
\]
Since
\[
\gamma \frac{p}{2 - p} = \gamma \frac{1 + \varepsilon}{1 - \varepsilon} = \delta < \frac{\pi}{2},
\]
Zygmund’s lemma yields that
\[
\left\| \varrho^{\frac{p}{2 - p}} \right\|_{1} = \left\| \exp \left( \frac{1 + \varepsilon}{1 - \varepsilon} H \varphi \right) \right\|_{1} \leq C_{1}.
\]
Consequently,
\[
\|\varrho H u\|_{p} \leq C_{1}^{\left(\frac{2-p}{2}\right)} \|\varrho^{\frac{1}{2}} H u\|_{2} \leq C_{1}^{\frac{1}{2}} \|\varrho^{\frac{1}{2}} H u\|_{2}. \tag{11}
\]
Applying now the theorem of Helson and Szegö to the weight $\varrho^{1/2} = \exp \left( \frac{1}{2} H \varphi \right)$, we obtain the estimate
\[
\|\varrho^{1/2} H u\|_{2} \leq C_{2} \|\varrho^{1/2} u\|_{2}. \tag{12}
\]
Again by Hölder’s inequality,
\[
\|\varrho^{1/2} u\|_{2}^{2} = \int \varrho |u|^{2} \, d\tau = \int \varrho^{-1} (\varrho |u|)^{2} \, d\tau \leq \left\{ \int \varrho^{-1-\varepsilon} \, d\tau \right\}^{1/(1+\varepsilon)} \left\{ \int (\varrho |u|)^{2(1+\varepsilon)} / \varepsilon \, d\tau \right\}^{\varepsilon/(1+\varepsilon)} = \|\varrho^{-1-\varepsilon}\|_{1}^{1/(1+\varepsilon)} \|\varrho u\|_{q}^{2}.
\]
Since $\varrho^{-1-\varepsilon} = \exp \left( ( -1 - \varepsilon ) \varphi \right)$ and $\| (1 + \varepsilon ) \varphi \|_{\infty} \leq \delta < \pi/2$, Zygmund’s inequality yields
\[
\|\varrho^{-1-\varepsilon}\|_{1}^{1/(1+\varepsilon)} \leq C_{1}^{1/(1+\varepsilon)} \leq C_{1},
\]
and hence
\[
\|\varrho^{1/2} u\|_{2} \leq C_{1}^{1/2} \|\varrho u\|_{q}. \tag{13}
\]
Combining the estimates (11), (12), and (13) we arrive at
\[
\|\varrho H u\|_{p} \leq C_{1}^{1/2} \|\varrho^{1/2} H u\|_{2} \leq C_{1}^{1/2} C_{2} \|\varrho^{1/2} u\|_{2} \leq C_{1} C_{2} \|\varrho u\|_{q}. \tag{14}
\]
In order to prove Lemma 4 we remark that the uniform boundedness of the derivative \( \partial_s F \) implies the uniform boundedness of \( a := \partial_s F(\cdot, u - u(0) + u_0) \) for all \( u \in W^1_p \).

**Lemma 6.** For each positive \( C_0 \) there exist numbers \( p > 1 \) and \( C > 0 \) such that any solution \( \tilde{w} = \tilde{u} + i\tilde{v} \) of the Riemann-Hilbert problem \( v - au = c \) with \( \|a\|_\infty \leq C_0 \) satisfies the estimate

\[
\|\tilde{w}\|_p \leq C \|c\|_\infty.
\]

**Proof.** The symbol \( 1 + ia \) of the Riemann-Hilbert problem \( v - au = c \) satisfies

\[
|\arg(1 + ia)| \leq \delta < \pi/2.
\]

The result then follows on applying the generalized Helson-Szegö theorem to the formulas in Theorem 13 which represent the solution. \( \Box \)

After these preparations we are in a position to prove Lemma 4. Since \( \|c\|_\infty \) is bounded by a constant that does not depend on the choice of \( u \), Lemma 6 yields the (uniform) boundedness of \( \|\tilde{u}\|_p \) in \( L^p \) for all \( p \) sufficiently close to 1. Integrating along \( \mathbb{T} \) then finally shows that \( Ku \) is bounded in \( W^1_p \).

**Proof of Theorem 18.**

1. The existence of a fixed point of \( K \) (and hence of a solution to the Riemann-Hilbert problem) in \( H^\infty \cap W^1_p \) for \( p \) close to 1 follows from Schauder’s fixed point theorem.

2. Since, for every \( p > 1 \), \( K \) maps \( W^1_p \) into \( W^1_q \) for each \( q \) with \( 1 \leq q < \infty \), any fixed point belongs to \( W^1_q \) for all finite \( q \).

3. Finally, if \( w_1 \) and \( w_2 \) are two solutions of (5) in \( H^\infty \cap C \), then their difference is a solution of a homogeneous linear Riemann-Hilbert problem

\[
v_2 - v_1 = f(\cdot, u_2) - f(\cdot, u_1) = g \cdot (u_2 - u_1)
\]

with continuous coefficients. This problem has index zero and since \( u_2(0) - u_1(0) = 0 \), we have \( w_1 = w_2 \).

Remark: The solution \( w \) depends continuously on the parameter \( u_0 \) in the side condition \( u(0) = u_0 \) and the real part of \( u \) tends uniformly to \( \pm \infty \) if \( u_0 \to \pm \infty \). Then a continuity argument shows that the Riemann-Hilbert problem \( v = F(\cdot, u) \) with the side condition \( u(1) = u_1 \) at \( t = 1 \) (or at any point on \( \mathbb{T} \)) has a (unique) solution.

4.2 Problems with compact target manifold

This and the next section are devoted to general nonlinear Riemann-Hilbert problems in implicit form

\[
F(t, u(t), v(t)) = 0.
\]

Instead of writing the boundary condition as an equation we prefer the geometric formulation

\[
w(t) \in M_t,
\]

23
where \( \{ M_t : t \in \mathbb{T} \} \) is a given family of target curves in the complex plane. To write the boundary condition in a more compact form, we introduce the set

\[
M := \{(t, z) \in \mathbb{T} \times \mathbb{C} : z \in M_t \},
\]

which is referred to as the target manifold of the problem, and the trace of a solution \( w \) as the graph of its boundary function,

\[
tr w := \{(t, w(t)) : t \in \mathbb{T} \}.
\]

The boundary condition can then be written simply as

\[
tr w \subset M.
\]

Here we consider the case where the target curves \( M_t \) are simple closed curves and the target manifold \( M \) is compact. A compact target manifold \( M \) is called admissible, if it is a submanifold of \( \mathbb{T} \times \mathbb{C} \) of smoothness class \( C^1 \) which is transverse to every plane \( \{t\} \times \mathbb{C} \). The figure below illustrates the construction of an admissible compact target manifold from its target curves and shows the trace of a solution.

In order to acquire a feeling for what may happen let us consider three examples.

**Example 1.** If \( M_t := \mathbb{T} \) for all \( t \in \mathbb{T} \), the solutions \( w \in H^\infty \cap C \) are exactly the finite Blaschke products:

\[
w(z) = c \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z}_k z},
\]

where \( z_k \in \mathbb{D} \) and \( c \in \mathbb{T} \) are arbitrarily chosen numbers.

**Example 2.** If \( M_t := \{ w \in \mathbb{C} : |w - t^{-1}| = 1 \} \), then \( w \equiv 0 \) is the only solution. This follows from the observation that any solution satisfies \( \text{Re} t w(t) \geq 0 \) on \( \mathbb{T} \) and \( z w(z) = 0 \) at \( z = 0 \), which is only possible if \( w \) vanishes throughout \( \overline{\mathbb{D}} \).

**Example 3.** If \( M_t := \{ w \in \mathbb{C} : |w - 2t^{-1}| = 1 \} \), then, by a similar reasoning (or simply by the argument principle) there is no holomorphic function such that \( w(t) \in M_t \) for all \( t \in \mathbb{T} \).
These examples stand for three general subclasses of Riemann-Hilbert problems with compact target manifold. We denote by int $M$ the bounded component of $(\mathbb{T} \times \mathbb{C}) \setminus M$ and by clos int $M$ its topological closure.

A target manifold $M$ is said to be regularly traceable if there exists a holomorphic function $w_M \in H^\infty \cap C$ with $\text{tr } w_M \subset \text{int } M$; it is called singularly traceable if it is not regularly traceable but there still exists $w_M \in H^\infty \cap C$ with $\text{tr } w_M \subset \text{clos int } M$; otherwise $M$ is called nontraceable.

The classes of regularly, singularly, or nontraceable target manifolds are denoted by $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{N}$, respectively.

The first of the above examples is of class $\mathcal{R}$, the second belongs to $\mathcal{S}$, and third is in $\mathcal{N}$, respectively. As we shall see in a moment, problems of class $\mathcal{R}$ have a solution set with a rich structure. The solutions can be classified by a geometric characteristic, the winding number $\text{wind}_M w$ about the target manifold.

In order to define this winding number, we fix an arbitrary complex-valued function $w_M \in C(\mathbb{T})$ with $\text{tr } w_M \subset \text{int } M$, and set

$$\text{wind}_M f := \text{wind } (f - w_M),$$

where the “wind” on the right-hand side refers to the usual winding number about the origin. This definition is independent of the choice of the function $w_M$.

The figures show a target manifold with traces of solutions having winding numbers zero and two, respectively.

The problem of conformally mapping $\mathbb{D}$ onto a simply connected $C^1$-region $G$ is another special case of a Riemann-Hilbert problem with a compact target manifold. It results from putting $M_t := \partial G$ for all $t \in \mathbb{T}$. Usually conformal mappings are required to be schlicht (i.e. one-to-one). In the context of Riemann-Hilbert problems this is reflected in the additional “winding condition” $\text{wind}_M w = 1$. We discuss the solvability of Riemann-Hilbert problems with regularly traceable target manifolds. The next theorem shows that the solution set has the same structure as for Example 1.

**Theorem 20.** Let $M \in \mathcal{R}$ with $w_M \in H^\infty \cap C$ satisfying $\text{tr } w_M \subset \text{int } M$. Then the following assertions hold.
(i) For arbitrarily given points \( t_0 \in \mathbb{T} \) and \( W_0 \in M_{t_0} \) there exists exactly one solution \( w \in H^\infty \cap C \) with \( \text{wind}_M w = 0 \) which satisfies the boundary condition
\[
\text{tr} \, w \subset M
\] (16)
and the interpolation condition
\[
w(t_0) = W_0.
\] (17)

(ii) Given any positive integer \( n \) and points \( t_0 \in \mathbb{T}, \, W_0 \in M_{t_0}, \, z_1, \ldots, z_n \in \mathbb{D} \), there exists exactly one solution \( w \in H^\infty \cap C \) with \( \text{wind}_M w = n \) which satisfies the conditions (16), (17), and the interpolation conditions
\[
w(z_j) = w_M(z_j), \quad j = 1, \ldots, n.
\] (18)

(iii) The target manifold \( M \) is covered in a schlicht manner by the traces of the solutions to (16) with winding number zero. The same result holds for the set of solutions with \( \text{wind}_M w = n \) which satisfy (18) for fixed numbers \( z_1, z_2, \ldots, z_n \).

(iv) The solutions described in (i) and (ii) are the only solutions of (16) in \( H^\infty \cap C \). In particular, there is no solution with a negative winding number about \( M \).

The figure shows a target manifold and traces of solutions with winding number zero.

For a proof of this result it is not sufficient to apply a simple fixed point theorem. All proofs known so far utilize homotopy methods. We give an outline of this approach. The reader interested in details of the proof is referred to [22].

The main idea is to imbed the given problem in a family of problems depending on a parameter \( \lambda \in [0, 1] \). This embedding depends continuously (in a definite sense) on the homotopy-parameter \( \lambda \). The homotopy is chosen so that it coincides with the given problem for \( \lambda = 1 \), and with a simple problem (where a solution is known) for \( \lambda = 0 \). Since we know that solutions exist for \( \lambda = 0 \) we just have to show that they cannot disappear along the homotopy when \( \lambda \) runs from 0 to 1. The theory of Leray-Schauder is a useful tool to settle this task. Its realization requires essentially three steps.
First of all one has to construct the homotopy. For Riemann-Hilbert problems with closed target curves the idea consists in deforming these curves to unit circles for \( \lambda = 0 \).

Secondly, the problems have to be reformulated as fixed point equations for compact operators \( K_\lambda \). This can be done in analogy to the approach used for explicit Riemann-Hilbert problems.

The third step is harder and quite technical. In order to ensure that the solutions ‘do not escape to infinity’ along the homotopy, we need an a-priori estimate. This means, assuming the existence of solutions \( w_\lambda \) corresponding to the homotopy-parameter \( \lambda \), we show that all solutions are uniformly bounded, so that \( \| w_\lambda \| < C \) for all \( \lambda \). This guarantees that all solutions (if such exist) remain in a certain ball \( B \).

According to Leray and Schauder we can associate with each operator \( K_\lambda \) an integer \( \text{fix}_B K_\lambda \), called its fixed-point index with respect to the ball \( B \). If this index is different from zero, then \( K_\lambda \) has a fixed point in \( B \). One main property of the fixed-point index is its homotopy invariance: If \( K_\lambda \) depends continuously on \( \lambda \) and for all \( \lambda \) no fixed point of \( K_\lambda \) lies on the boundary of \( B \), which is ensured by the a-priori estimate, then \( \text{fix}_B K_\lambda = \text{const} \). Especially, \( \text{fix}_B K_1 = \text{fix}_B K_0 \), so that \( K_1 \) has a fixed point, if we can show that \( \text{fix}_B K_0 \neq 0 \) for the simple operator \( K_0 \).

Riemann-Hilbert problems with singularly traceable target manifolds have completely different solvability properties.

**Theorem 21.** Let \( M \in S \) and suppose the function \( w_M \in H^\infty \cap C \) satisfies \( \text{tr} w_M \subset \text{clos} \text{int} M \). Then the following assertions hold.

(i) The function \( w_M \) is the only solution in \( H^\infty \cap C \) of the Riemann-Hilbert problem \( \text{tr} w \subset M \).

(ii) The winding number of \( w_M \) about \( M \) is negative.

The remaining case of non(trace)able target manifolds is trivial. We formulate the result only for the sake of completeness.

**Theorem 22.** Let \( M \in N \). Then there is no solution \( w \in H^\infty \cap C \) of the Riemann-Hilbert problem \( \text{tr} w \subset M \).

The class \( M \) of all admissible compact target manifolds can be endowed with a natural topology. Then it turns out that \( R \) and \( N \) are open sets and \( S \) is their common boundary. Thus singular problems are ‘degenerate’, and “generically” Riemann-Hilbert problems belong either to \( R \) or to \( N \). For details see [22].

### 4.3 Problems with noncompact target manifold

A prominent example of a Riemann-Hilbert problem with noncompact target manifold is given by linear problems with continuously differentiable coefficients and right-hand
side. More generally, we now assume that the target manifold satisfies the following conditions.

A non-compact orientable target manifold \( M \) is said to be \textit{admissible}, if it has a parametrization

\[
M = \{(t, \mu(t, s)) : t \in \mathbb{T}, s \in \mathbb{R} \}
\]  

(19)

where \( \mu \in C^1(\mathbb{T} \times \mathbb{R}) \) is such that

(i) for each \( t \in \mathbb{T} \) the mapping \( s \mapsto \mu(t, s) \) is injective,

(ii) \( 0 < 1/C \leq |D_s \mu| \leq C \) on \( \mathbb{T} \times \mathbb{R} \),

(iii) the limits

\[
\lim_{s \to \pm \infty} D_s \mu(t, s) := \mu^\pm(t), \quad \lim_{s \to \pm \infty} s^{-2} D_t \mu(t, s) = 0
\]

exist uniformly with respect to \( t \in \mathbb{T} \).

The number \( \text{ind} M := \text{wind} \mu^+ \) is said to be the \textit{index} of the problem. Geometrically, the index is the ‘twisting number’ of the target manifold \( M \). For the time being, all admissible target manifolds are orientable by assumption.

Condition (iii) guarantees a certain regular behavior of the target curves \( M_t \) at infinity. While the condition implies that the direction of the target curves stabilizes for \( s \to \pm \infty \), the existence of a (linear) asymptotics does not follow. Note that there is no interrelation between the behavior at 'both ends' of \( M_t \).

The figure shows (part of) two target manifolds with index zero and index two.

The qualitative results for nonlinear problems with noncompact orientable target manifold are completely analogous to the linear case.

\textbf{Theorem 23.} Let \( M \) be a noncompact admissible target manifold. Then the following assertions hold.

(i) If \( \text{ind} M = 0 \) the target manifold is covered by the graphs of solutions \( w \in H^\infty \cap C \) to the Riemann-Hilbert problem \( \text{graph} w \subset M \) in a schlicht manner.

(ii) If \( \text{ind} M < 0 \) there exists at most one solution \( w \in H^\infty \cap C \) of \( \text{graph} w \subset M \).
If \( \text{ind } M = n \geq 0 \) then, for arbitrarily given \( t_0 \in \mathbb{T}, W_0 \in M_{t_0}, z_1, \ldots, z_n \in \mathbb{D}, W_1, \ldots, W_n \in \mathbb{C} \), there exists exactly one solution of graph \( w \subset M \) which satisfies the interpolation conditions

\[
w(t_0) = W_0, \quad w(z_k) = W_k \quad (k = 1, \ldots, n).
\] (20)

The case of nonorientable noncompact target manifolds can be dealt with by considering the doubled manifold \( \tilde{M} \) having the fibers \( \tilde{M}_t := M_{t^2} \). It is remarkable that Riemann-Hilbert problems with a unique solution are exactly those with a nonorientable manifold with \( \text{ind } M = -1/2 \). The reader interested in the details may consult [22].

5 Applications

5.1 Boundary value problems for harmonic functions

There are several boundary value problems for harmonic functions (in plane domains) which can be reformulated as Riemann-Hilbert problems. Since harmonic functions are invariant under conformal mapping, we restrict ourselves to problems on the disk. Because any harmonic function \( u \) in the disk has a harmonic conjugate \( v \) such that \( w = u + iv \) is holomorphic, the Dirichlet problem

\[
\Delta u(z) = 0 \quad \text{in } \mathbb{D}, \quad u(t) = f(t) \quad \text{on } \mathbb{T}
\]

is (more or less) equivalent to the Riemann-Hilbert problem \( \text{Re } w = f \). Of course, the Dirichlet problem can be solved explicitly by Poisson’s formula.

More interesting problems result from the observation that for harmonic \( u \) the function \( w := \partial_x u - i\partial_y u \) is holomorphic. So any problem

\[
\Delta u(z) = 0 \quad \text{in } \mathbb{D}, \quad F(t, \partial_x u(t), \partial_y u(t)) = 0 \quad \text{on } \mathbb{T}
\]

with boundary condition involving the gradient of \( u \) (but not \( u \) itself) can be reformulated as a Riemann-Hilbert problem. In the special case where \( F \) is linear with respect to \( u \) and \( v \) we obtain the oblique derivative problem

\[
\Delta u(z) = 0 \quad \text{in } \mathbb{D}, \quad a(t) \partial_x u(t) + b(t)\partial_y u(t) = c(t) \quad \text{on } \mathbb{T}
\]

which generalizes the well-known Neumann problem. The related linear RHP has index \( \text{wind}(b + ia) \), which is always \(-1\) for the Neumann problem (why?). In the next section we give an example of a nonlinear oblique derivative problem in hydrodynamics.

5.2 A problem in hydrodynamics

It is well known that conformal mapping techniques are useful in studying flow problems around obstacles. Here we consider a situation where the fluid can penetrate the
boundary of the obstacle. More precisely, we investigate the plane steady irrotational flow of an incompressible inviscid fluid with density $\varrho$ past a (not necessarily circular) cylinder with permeable (porous or perforated) surface.

Let $G$ denote the intersection of the inner region of the cylinder with the $\zeta$-plane ($\zeta = \xi + i\eta$) orthogonal to the axis of the cylinder. We look for the pressure field $p$ and for the velocity field $\vec{v} = (v_\xi, v_\eta)$. As usual the influence of gravity is neglected.

In the outer region we assume that the flow at infinity is parallel to the $\xi$-axis and has the speed $q_\infty$,

$$
v_\xi(\infty) = q_\infty, \quad v_\eta(\infty) = 0.
$$

The pressure at infinity is denoted by $p_\infty$. The velocity components of the flow in the direction of the (positively oriented) tangent and the inner normal on the boundary $\partial G$ are denoted by $v_\sigma$ and $v_\nu$, respectively. The circulation $J$ of the flow around $\partial G$ and the flux $Q$ through $\partial G$ are given by

$$
J := \int_{\partial G} v_\sigma \, d\sigma, \quad Q := \int_{\partial G} v_\nu \, d\sigma,
$$

where integration is with respect to arc length $\sigma$ on $\partial G$. Notice that $Q > 0$ means suction of liquid out of the flow and $Q < 0$ injection of liquid into the flow.

The assumption that the flow be irrotational in the outer region of the cylinder and the continuity equation guarantee that the complex velocity $W := v_\xi - i v_\eta$ is a holomorphic function in the outer domain $\mathbb{C} \setminus \overline{G}$. Because of (21) and (22) its asymptotic expansion at infinity is

$$
W(\zeta) = q_\infty - \frac{Q + i J}{2\pi} \frac{1}{\zeta} + O(\zeta^{-2}).
$$

The filtration process through the boundary of the cylinder is described by a filtration law. Here we assume that the filtration velocity is equal to the normal component $v_\nu$ and that is a continuous monotone function of the difference $p - p_i$ of the pressure at the outer and the inner surface of the cylinder.
We admit that the filtration law depends on the point \( s \in \partial G \),
\[
v_\nu(s) = \Phi(s, p(s) - p_i(s)).
\]

The case of a linear law \( \Phi(s, p) = K(s) p \) is of particular interest.
The simplest model for the interior of the cylinder is the assumption of constant pressure \( p_0 \).
From the physical (and mathematical) point of view we get three reasonable problems
if any two of the three quantities, pressure \( p_0 \) inside the cylinder, total flux \( Q \) through
the surface, and circulation \( J \) around the cylinder are prescribed and the third one is
free. Here we consider the case where \( Q \) is unknown.
The Bernoulli equation in the outer region allows to eliminate the pressure \( p \) from the
speed \( q \),
\[
p = p_\infty + \frac{\rho}{2} q_\infty^2 - \frac{\rho}{2} q^2.
\]
On the outer surface of the cylinder we have \( q^2 = v_\nu^2 + v_\sigma^2 \), and inserting this into the
filtration law we get the basic boundary relation
\[
v_\nu = \Phi\left(., d - \frac{\rho}{2} (v_\nu^2 + v_\sigma^2)\right), \tag{24}
\]
with the pressure constant
\[
d := \frac{\rho}{2} q_\infty + p_\infty - p_0.
\]
This is a nonlinear oblique derivative problem for the velocity potential \( \Psi \), which is a
harmonic function in the flow region with \( \text{grad} \Psi = (v_\xi, v_\eta) \). In order to transform it
to a Riemann-Hilbert problem in the disk we use a conformal mapping \( \varphi \) of \( \mathbb{C} \setminus \overline{G} \) onto
the exterior \( \mathbb{C} \setminus \overline{aD} \) of a disk, normalized at infinity by \( \varphi(\zeta) = \zeta + O(1) \). The positive
number \( a \) is uniquely determined.
On account of (23) and the normalization of \( \varphi \), the function \( \tilde{W} := (W \circ \varphi^{-1}) / (\varphi' \circ \varphi^{-1}) \)
is holomorphic in \( \mathbb{C} \setminus \overline{aD} \) and has the expansion
\[
\tilde{W}(Z) = q_\infty - \frac{Q + iJ}{2\pi} \frac{1}{Z} + O(Z^{-2}) \tag{25}
\]
at infinity. The normal and tangential velocity components \( v_\nu \) and \( v_\sigma \) on \( \partial G \) are
transformed according to
\[
v_\nu + iv_\sigma = W \cdot (n_\xi + in_\eta) = \tilde{W} \cdot \varphi' \cdot (n_\xi + in_\eta) = \tilde{W} \cdot |\varphi'| \cdot (n_X + in_Y),
\]
where \( n_\xi, n_\eta \) and \( n_X, n_Y \) are the components of the inner normal vectors to \( \partial G \) and
to \( aT \) at the points \( \zeta \) and \( Z = \varphi(\zeta) \), respectively. Hence, putting \( \tilde{W} = \tilde{U} + i\tilde{V} \)
and separating the real and imaginary parts, we obtain that
\[
v_\nu = \Lambda(-\tilde{U} \cos \tau + \tilde{V} \sin \tau), \quad v_\sigma = \Lambda(-\tilde{U} \sin \tau - \tilde{V} \cos \tau).
\]
Here $\Lambda := |\varphi'|$ is the boundary distortion of the conformal mapping $\varphi$, and $\tau$ denotes the polar angle of the point $Z$. By the definition of $\tilde{W}$, we have on the contour
\[ q^2 = U^2 + V^2 = \Lambda^2(\tilde{U}^2 + \tilde{V}^2). \] (26)
By virtue of the relations (25)–(26), the boundary condition (24) takes the form
\[ \Lambda(-\tilde{U}\cos\tau + \tilde{V}\sin\tau) = \Phi\big(.,d - \Lambda^2\frac{\partial}{2}(\tilde{U}^2 + \tilde{V}^2)\big). \]
After the transformation $z = a/Z$ we arrive at a Riemann-Hilbert problem for a function $\tilde{w}(z) = \tilde{u}(z) + i\tilde{v}(z) := \tilde{W}(a/z)$, which is holomorphic in the unit disk. The boundary condition now reads
\[ \tilde{u}\cos\tau + \tilde{v}\sin\tau = -\frac{1}{\Lambda}\Phi\big(.,d - \Lambda^2\frac{\partial}{2}(\tilde{u}^2 + \tilde{v}^2)\big), \]
where $\tilde{u} = \tilde{u}(e^{i\tau})$, $\tilde{v} = \tilde{v}(e^{i\tau})$. The asymptotic behavior of $\tilde{W}$ at infinity implies that
\[ \tilde{w}(z) = q_\infty - \frac{Q + iJ}{2\pi a} z + O(z^2) \]
in a neighborhood of the origin. The ansatz $\tilde{w}(z) = q_\infty(1 + z w(z))$, with a new holomorphic function $w$, and a short calculation transform the boundary condition into a dimension-less form,
\[ u + \cos\tau = -F\big(.,c - E^2(u + \cos\tau)^2 - E^2(v - \sin\tau)^2\big), \] (27)
and the additional condition becomes
\[ w(0) = -\frac{Q + iJ}{2\pi a q_\infty}. \] (28)
Here we used the abbreviations
\[ F(t,p) := \Phi\big(s, p q_\infty^2\big)/(q_\infty \Lambda(s)), \quad E(t) := \Lambda(s)/\sqrt{2}, \quad c := (p_\infty - p_0)/(q_\infty^2) + 1/2, \]
with $s := \varphi^{-1}(at)$. So the filtration problem has the following formulation as a Riemann-Hilbert problem in the disk:
To given real numbers $c$ and $J$ find a function $w = u + iv$ holomorphic in the unit disk and a real constant $Q$ such that the boundary condition (27) and the side condition (28) are satisfied.
Note that only $J$ is prescribed, while $Q$ has to be determined, so that condition (28) fixes the imaginary part of $w(0)$ only. After solving the problem, $Q$ can be computed from the real part of $w(0)$.
The geometry of the target curves depends significantly on the growth of the filtration law for $p \to -\infty$. For laws with $\Phi(s,p) \sim |p|^\kappa$ and $\kappa > 1/2$ we typically obtain closed
target curves, while $\kappa < 1/2$ corresponds to non-closed curves. In the linear case the
target curves are circles. If the filtration constant $K$ is constant the problem can then
even be solved explicitly.

The Riemann-Hilbert problem under discussion has solutions with different winding
numbers. The physically relevant solutions have winding number zero. They are the
only solutions which reproduce the flow around a solid cylinder as the function $\Phi$ in
the filtration law tends to zero.

It should finally be mentioned that the Riemann-Hilbert approach reduces the un-
bounded exterior flow problem to a problem on the boundary of the region. In par-
ticular, for computational purposes, this is an advantage over domain discretization
methods, since it reduces a two–dimensional problem in an unbounded domain to a
problem on a bounded one–dimensional manifold.

The figure shows the streamlines and the pressure distribution (color) in a flow with
$J = 0$ and $J > 0$.

5.3 Design of dynamical systems

Another question which is intimately (but not obviously) related to Riemann-Hilbert
problems was raised in the eighties by J. W. Helton. Its engineering background is
design of dynamical systems via optimization of the frequency response function.

Roughly speaking, a dynamical system takes input functions and produces output func-
tions. Usually these functions live on the real line and are considered as time dependent
signals.

The most important dynamical systems are causal, linear, time invariant and stable
systems. Those systems are completely characterized by their transfer function $T$; the
Fourier transform $F$ of the input signal and $G$ of the output signal are related by

$$G(\omega) = T(\omega) \cdot F(\omega).$$

It follows from Payley–Wiener theory that the transfer function of causal systems
extends to a bounded holomorphic function in the upper half–plane (respectively to
the right half-plane, if one works with the Laplace transform, which is frequently done in the engineering literature).

A system operating at a harmonic input \( \exp(i\omega t) \) (here \( t \in \mathbb{R} \) denotes time) with frequency \( \omega \) produces the output \( T(\omega) \cdot \exp(i\omega t) \), which has the same frequency as the input, but different amplitude and phase. The values \( |T(\omega)| \) and \( \arg T(\omega) \) describe gain and phase shift of the system at frequency \( \omega \). For this reason the boundary function \( f \) of \( T \) on the real axis is called the frequency response of the system, \( f(\omega) := T(\omega) \).

Since \( T \) is a bounded holomorphic function it can be reconstructed from its boundary function \( f \). Conversely, any bounded holomorphic function \( T \) in the right half plane is the transfer function of such a system.

*Linear, causal, time invariant, stable systems are completely determined by its frequency response.*

So much for the description – but equally important is the design of systems. Usually a designer knows what he would consider to be a ‘good performance’ \( f(\omega) \) of the system at frequency \( \omega \). However, the frequency response is the boundary function of a holomorphic function, so it cannot be prescribed arbitrarily; design of systems is an optimization problem.

Here we assume that the performance of a (virtual) system with frequency response \( f \) is evaluated in terms of a penalty function

\[
F: \mathbb{R} \times \mathbb{C} \to \mathbb{R}_+, \ (\omega, w) \mapsto F(\omega, w),
\]

where \( F(\omega, w) \) attains small values if \( w = f(\omega) \) is ‘good’ and large values if it is ‘bad’. The standard choice of \( F \) is \( F(\omega, w) := |w - f_0(\omega)| \), the euclidean distance from an ‘ideal’ response \( f_0 \). The figure shows the level sets (color) of a more general penalty function.

Once the penalty function \( F \) is fixed, the over–all rating of a system, in the sense of worst case analysis over all frequencies, is the real number

\[
P(f) := \sup \{ F(\omega, f(\omega)) : \omega \in \mathbb{R} \}
\]

Optimal design of a system then requires to find the optimal performance

\[
P^* := \inf \{ P(f) : f \text{ is a frequency response function} \}
\]
and all systems (if such exist) for which the infimum is attained.
In order to relate the optimization problem to Riemann-Hilbert problems we transform
the transfer functions from the right half plane to the unit disk (a related conformal
mapping is the Cayley transform) and introduce the sets of bounded analytic functions
with restricted boundary values

\[ A(p) := \{ w \in H^\infty(D) : F(t, w(t)) \leq p \text{ a.e. on } \mathbb{T} \}. \]

The set \( A(p) \) is built from the transfer functions of all systems with performance not
worse than \( p \), and the optimization problem is equivalent to find the smallest possible
value of \( p \) for which \( A(p) \) is not empty.
The left figure shows traces of functions in \( A \), the right figure depicts the nested level
sets

\[ M(p) := \{(t, w) \in \mathbb{T} \times \mathbb{C} : F(t, w) = p\} \]

for several values of \( p \).

It turns out that the properties of \( A(p) \) are closely related to the Riemann-Hilbert
problem with the target manifold \( M(p) \).

**Theorem 24.** Let \( M(p) \) is an admissible target manifold and denote by \( \#A(p) \) the
number of elements in \( A(p) \). Then \( \#A(p) > 1, \#A(p) = 1, \) or \( \#A(p) = 0 \), if and only
if \( M(p) \in \mathcal{R}, M(p) \in \mathcal{S}, \) or \( M(p) \in \mathcal{N}, \) respectively.

The result is much deeper than it looks like. For instance it implies that any \( A(p) \)
which contains a bounded function \( w \in H^\infty \) automatically also contains a function in
the disk algebra \( H^\infty \cap \mathbb{C} \).

Returning to the optimization problem we consider strictly nested families of target
manifolds \( M(p) \), which means that the assumptions (i)–(iii) are satisfied:

(i) The \( M(p) \) are admissible compact target manifolds, depending continuously on
\( p \in \mathbb{R}_+ \).

(ii) For all \( p \) and \( q \) with \( 0 < p < q \) we have \( M(p) \subset \text{int } M(q) \).

(iii) \( M(p) \in \mathcal{R} \) if \( p \) is sufficiently large; \( M(p) \in \mathcal{N} \) if \( p \) is sufficiently small.
Theorem 25. For any strictly nested family of target manifolds the optimization problem has a unique solution. A function \( w \in H^\infty \) is a solution of the optimization problem if and only if

\[
w \in H^\infty \cap C, \quad F(t, w(t)) = p^* = \text{const} \quad \text{on } \mathbb{T}, \quad \text{wind} \partial_w F(\cdot, w) \geq 1.
\]

Here \( \partial_w \) denotes the antiholomorphic Cauchy-Riemann operator \( \partial_w = \partial_u - i \partial_v \). Together with \( F(t, w(t)) = p^* \) this yields \( \text{wind}_{M(p^*)} w = -\partial_w F(\cdot, w) = -k < 0 \). For generic problems we have \( k = 1 \), which implies that \( k > 1 \) is ‘very unlikely’.

Assertion (ii) reflects the fact that the solution of the optimization problem is the solution of a singular Riemann-Hilbert problem. It has the special meaning that optimal systems ‘flatten the penalty’, i.e., they have one and the same performance for all frequencies.

It is an important feature of this approach that the optimization problem is transformed into a (parameter depending) boundary value problem; we get equations instead of inequalities. This allows, for instance, the construction of efficient numerical methods for solving the design problem.

6 Historical Remarks

The history of the Riemann-Hilbert problem can be traced back to the origins of function theory. The problem enters the scene in 1851, the year of Riemann’s famous thesis “Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse” (see [16], pp. 3–47). When studying conditions which uniquely define a function holomorphic in a domain, Riemann wrote: “Die Bedingungen, welche soeben zur Bestimmung der Function hinreichend und nothwendig befunden worden sind, beziehen sich auf ihren Werth in Begrenzungspunkten ... und zwar geben sie für jeden Begrenzungspunkt eine Bedingungsgleichung. Bezieht sich jede derselben nur auf einen Begrenzungspunkt, so werden sie durch eine Schar von Kurven repräsentirt, von denen für jeden Begrenzungspunkt eine den geometrischen Ort bildet.”

Not regarding the connection with the problem of conformal mapping, Riemann himself treated the problem only heuristically. The first one who proposed a method appropriate for solving the linear problem was David Hilbert. In a lecture given at the 3rd International Mathematical Congress in Heidelberg in 1904 he introduced the singular integral operator with cotangent kernel and transformed the problem to a Fredholm integral equation with a continuous kernel. Later on Hilbert published a modification of his earlier approach and made a few remarks about the index which he disregarded formerly, as was pointed out by Fritz Noether [15] in 1921. Noether’s courageous work even contains material on problems with piecewise continuous coefficients. Fritz Noether’s paper [15] about linear Riemann-Hilbert problems and related singular integral equations is without doubt one of the milestones of functional analysis – it was this paper where the index of an operator appeared for the first time.
Noether’s discovery was the starting point of a rapidly developing theory which relates geometric properties (here: the winding number of the function $f$) to solvability properties of an operator equation (dimension of kernel and codimension of the range). It led subsequently to a general symbol concept, being introduced by Solomon Mikhlin in the late thirties, and culminated in Lars Hörmander’s theory of pseudodifferential operators.

During the ‘classical period’ linear Riemann-Hilbert problems and related singular integral equations with Hölder continuous coefficients where studied by F. D. Gakhov, N. I. Muskhelishvili, I.N. Vekua, and others. The scalar linear Riemann-Hilbert problem with measurable coefficients could finally be treated by I. B. Simonenko [19] in 1960. Beginning in the sixties, Banach algebra techniques were developed, which allowed to attack some of the remaining hard questions (cf. Gohberg and Krupnik [7]). Combined with so-called local principles, they form the heart of the modern theory of Riemann-Hilbert problems, singular integral equations and Toeplitz operators. This is, however already another story. The reader who is willing to learn more about those developments is referred to the monograph by Böttcher and Silbermann [1].

Riemann-Hilbert problems with small nonlinearities where already studied in the fifties by Gekht, Gusejnov, Natalevich, and others, using the fixed point theorems of Banach and Schauder.

Without assuming that the nonlinearities are “sufficiently small”, L. v. Wolfersdorf obtained solvability results by an application of Schauder’s fixed point theorem. The material on explicit problems is directly related to [25]. The proof given there still requires an additional “oscillation condition”, which could be omitted in [20]. The trick of differentiating the boundary condition is taken from [26]. Explicit problems with globally Lipschitz continuous function $F$ have been studied by Efendiev and Wegert [24].

For general classes of nonlinear Riemann-Hilbert problems the real breakthrough came in 1972 with a paper by A.I. Shnirelman [18]. He proved the solvability of regularly traceable Riemann-Hilbert problems with target manifold of smoothness class $C^2$. Shnirel’man’s original proof [18] was based on a specific topological degree theory for so-called quasi–linearlike mappings. Later L.v.Wolfersdorf invented the ‘differentiation trick’ which allowed to work with the simpler concept of Leray-Schauder degree. It also allowed to relax the smoothness assumptions to $C^1$, see Wegert [20]. An alternative $C^2$–approach, using the implicit function theorem in Banach spaces, was proposed by Forstnerič [5]. Problems with non-compact target manifold have been treated in full generality for the first time in [20].

For a comprehensive treatment of nonlinear Riemann-Hilbert problems with compact and non-compact target manifolds, and a number of applications, we refer to [22]. A survey on later developments up to the end of the last century is in [23].

The general $H^\infty$–optimization problem in the design of dynamical systems was introduced by J.W. Helton in the eighties (see [9]). Though the ‘flattening property’ of continuous solutions is known since 1986 (Helton and Howe [10]), the existence of
continuous solutions has been open for several years. It was shown independently by Helton and Marshall [11], and, in a more general setting, by Wegert [21], using the approach sketched above.

References


