

Höhere Mathematik für technische Studiengänge

Vorbereitungsaufgaben für die Übungen

Integralrechnung für eine Veränderliche

1. Lösungshinweise:

(Integrationskonstanten sind weggelassen.)

$$a) \int \frac{\ln x}{x} dx = \int t dt = \frac{t^2}{2} = \frac{(\ln x)^2}{2}$$

$$b) \int \frac{\cos x}{\sin^2 x} dx = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\sin x}$$

$$c) \int \frac{x}{1+x^4} dx = \frac{1}{2} \int \frac{dt}{1+t^2} = \frac{\arctan t}{2} = \frac{\arctan x^2}{2}$$

$$d) \int 3\sqrt{7x-5} dx = \frac{6}{7} \int t^2 dt = \frac{2}{7} t^3 = \frac{2}{7} (\sqrt{7x-5})^3$$

$$e) \int \cosh^3 x \sinh x dx = \int t^3 dt = \frac{t^4}{4} = \frac{\cosh^4 x}{4}$$

$$f) \int \frac{dx}{\cos^2(4x+7)} = \frac{1}{4} \int \frac{dt}{\cos^2 t} = \frac{\tan t}{4} = \frac{\tan(4x+7)}{4}$$

$$g) \int \frac{dx}{\sqrt{1-(3x^2)}} = \frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{3}} \arcsin t = \frac{\arcsin(\sqrt{3}x)}{\sqrt{3}}$$

$$h) \int x\sqrt{x^2+1} dx = \frac{1}{2} \int \sqrt{t} dt = \frac{1}{3} \sqrt{t^3} = \frac{1}{3} (x^2+1)\sqrt{x^2+1}$$

$$i) \int \frac{\sin x}{\sqrt{5+\cos x}} dx = -\int \frac{dt}{\sqrt{t}} = -2\sqrt{t} = -2\sqrt{5+\cos x}$$

$$\ln x = t, \quad \frac{dx}{x} = dt$$

$$\sin x = t, \quad \cos x dx = dt$$

$$x^2 = t, \quad 2x dx = dt$$

$$\sqrt{7x-5} = t, \quad \frac{7}{2\sqrt{7x-5}} dx = dt$$

$$\cosh x = t, \quad \sinh x dx = dt$$

$$4x+7 = t, \quad 4 dx = dt$$

$$\sqrt{3}x = t, \quad \sqrt{3} dx = dt$$

$$x^2+1 = t, \quad 2x dx = dt$$

$$5+\cos x = t, \quad -\sin x dx = dt$$

2. Lösungshinweise:

(Integrationskonstanten sind weggelassen.)

$$\int u \cdot v' dx = u \cdot v - \int u' \cdot v dx$$

$$a) \int x \cos x dx = x \sin x - \int 1 \cdot \sin x dx = x \sin x + \cos x$$

$$b) \int x^2 \sin x dx = x^2(-\cos x) - \int 2x(-\cos x) dx = -x^2 \cos x + 2 \int x \cos x dx = -x^2 \cos x + 2x \sin x + 2 \cos x$$

$$c) \int x e^{3x} dx = x \frac{1}{3} e^{3x} - \int 1 \cdot \frac{1}{3} e^{3x} dx = \frac{x}{3} e^{3x} - \frac{1}{9} e^{3x}$$

$$d) \int \arctan x dx = (\arctan x) x - \int \frac{1}{1+x^2} \cdot x dx = x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2)$$

$$e) \int \frac{\ln x}{\sqrt{x}} dx = (\ln x) 2x^{\frac{1}{2}} - \int \frac{1}{x} \cdot 2x^{\frac{1}{2}} dx = 2\sqrt{x} \ln x - 2 \int x^{-\frac{1}{2}} dx = 2\sqrt{x} \ln x - 4\sqrt{x} = 2\sqrt{x} (\ln x - 2)$$

$$u = x, \quad v' = \cos x$$

$$u = x^2, \quad v' = \sin x$$

$$u = x, \quad v' = \cos x$$

(vergl. Aufgabe a)

$$u = x, \quad v' = e^{3x}$$

$$u = \arctan x, \quad v' = 1$$

$$\text{Typ: } \int \frac{f'}{f} dx = \ln |f|$$

$$u = \ln x, \quad v' = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

<p>f) $\int (\ln x)^2 dx = (\ln x)^2 x - \int \frac{2 \ln x}{x} \cdot x dx$ $= x (\ln x)^2 - 2 \int \ln x dx$ $= x (\ln x)^2 - 2((\ln x) x - \int \frac{1}{x} \cdot x dx)$ $= x (\ln x)^2 - 2x \ln x + 2x$</p>	<p>$u = (\ln x)^2, v' = 1$ $u = \ln x, v' = 1$</p>
<p>g) $\int e^{-x} \sin x dx = e^{-x}(-\cos x) - \int (-e^{-x}) \cdot (-\cos x) dx$ $= -e^{-x} \cos x - \int e^{-x} \cos x dx$ $= -e^{-x} \cos x - (+e^{-x} \sin x - \int -e^{-x} \sin x dx)$ $= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x dx$ $2 \int e^{-x} \sin x dx = -e^{-x} \cos x - e^{-x} \sin x$ $\int e^{-x} \sin x dx = -\frac{1}{2} e^{-x} (\sin x + \cos x)$</p>	<p>$u = e^{-x}, v' = \sin x$ $u = e^{-x}, v' = \cos x$</p>
<p>h) $\int \sin^2 x dx = \sin x(-\cos x) - \int \cos x \cdot (-\cos x) dx$ $= -\sin x \cos x + \int \cos^2 x dx$ $= -\sin x \cos x + \int (1 - \sin^2 x) dx$ $2 \int \sin^2 x dx = -\sin x \cos x + x$ $\int \sin^2 x dx = \frac{1}{2}(x - \sin x \cos x)$ $= \frac{1}{4}(2x - \sin 2x)$</p>	<p>$u = \sin x, v' = \sin x$ $\sin^2 x + \cos^2 x = 1$ $\sin 2x = 2 \sin x \cos x$</p>
<p>i) $\int \cosh^2 x dx = \cosh x \sinh x - \int \sinh^2 x dx$ $= \cosh x \sinh x - \int (\cosh^2 x - 1) dx$ $2 \int \cosh^2 x dx = \cosh x \sinh x + x$ $\int \cosh^2 x dx = \frac{1}{2}(x + \sinh x \cosh x)$ $= \frac{1}{4}(2x + \sinh 2x)$</p>	<p>$u = \cosh x, v' = \cosh x$ $\cosh^2 x - \sinh^2 x = 1$ $\sinh 2x = 2 \sinh x \cosh x$</p>

3. Lösungshinweise:

(Integrationskonstanten sind weggelassen.)

a) Ansatz: $\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-4} = \frac{A(x^2-x-12)+B(x^2-5x+4)+C(x^2+2x-3)}{(x-1)(x+3)(x-4)}$

Koeffizientenvergleich:

$$2x^2 + 41x - 91 = (A + B + C)x^2 + (-A - 5B + 2C)x + (-12A + 4B - 3C)$$

$$\Rightarrow \begin{cases} A + B + C = 2 \\ -A - 5B + 2C = 41 \\ -12A + 4B - 3C = -91 \end{cases} \Rightarrow A = 4, B = -7, C = 5$$

Integration:

$$\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} dx = \int \left(\frac{4}{x-1} + \frac{-7}{x+3} + \frac{5}{x-4} \right) dx = 4 \ln |x-1| - 7 \ln |x+3| + 5 \ln |x-4|$$

b) Ansatz:

$$\frac{9x^2-2x+9}{(x-1)^2(x^2+2x+5)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+5} = \frac{A(x^3+x^2+3x-5)+B(x^2+2x+5)+C(x^3-2x^2+x)+D(x^2-2x+1)}{(x-1)^2(x^2+2x+5)}$$

Koeffizientenvergleich:

$$9x^2 - 2x + 9 = (A + C)x^3 + (A + B - 2C + D)x^2 + (3A + 2B + C - 2D)x + (-5A + 5B + D)$$

$$\Rightarrow \begin{cases} A + C = 0 \\ A + B - 2C + D = 9 \\ 3A + 2B + C - 2D = -2 \\ -5A + 5B + D = 9 \end{cases} \Rightarrow A = 1, B = 2, C = -1, D = 4$$

Integration: $\int \frac{9x^2-2x+9}{(x-1)^2(x^2+2x+5)} dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-x+4}{x^2+2x+5} \right) dx$
 $= \ln |x-1| - \frac{2}{x-1} - \frac{1}{2} \ln (x^2 + 2x + 5) + \frac{5}{2} \arctan \frac{x+1}{2}$

$$\begin{aligned} \text{c) } \int \frac{x+2}{x^3-2x^2+x} dx &= \int \frac{x+2}{x(x-1)^2} dx = \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \right) dx = \int \left(\frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2} \right) dx \\ &= 2 \ln |x| - 2 \ln |x-1| - \frac{3}{x-1} \end{aligned}$$

$$\begin{aligned} \text{d) } \int \frac{x^2-6}{(x-1)^3} dx &= \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \right) dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-5}{(x-1)^3} \right) dx \\ &= \ln |x-1| - \frac{2}{x-1} + \frac{5}{2(x-1)^2} \end{aligned}$$

$$\begin{aligned} \text{e) } \int \frac{4x^2+3x+10}{x^3+2x^2+5x} dx &= \int \frac{4x^2+3x+10}{x(x^2+2x+5)} dx = \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+2x+5} \right) dx = \int \left(\frac{2}{x} + \frac{2x-1}{x^2+2x+5} \right) dx \\ &= 2 \ln |x| + \ln(x^2+2x+5) - \frac{3}{2} \arctan \frac{x+1}{2} \end{aligned}$$

Bei Aufgaben f) - i) zuerst Abspalten des ganzrationalen Anteils durch Polynomdivision!

f) Integrand ist unecht gebrochene rationale Funktion, Grad des Zählerpolynoms größer als Grad des Nennerpolynoms!

Abspalten des ganzrationalen Anteils durch Polynomdivision:

$$(-2x^5 + 9x^3 + 4x^2 - 4x - 7) : (x^3 - 3x - 2) = -2x^2 + 3 + \frac{5x-1}{x^3-3x-2}$$

$$\text{Integration des ganzrationalen Anteil: } \int -2x^2 + 3 dx = -\frac{2}{3}x^3 + 3x$$

Partialbruchzerlegung des echt gebrochenen Anteils:

Nullstellen des Nenners:

- Erste Nullstelle $x_1 = -1$ durch „Probieren“ mit Teilern des Absolutgliedes -2 :

- Abspalten des Linearfaktors $(x - x_1)$ durch Polynomdivision $(x^3 - 3x - 2) : (x + 1) = x^2 - x - 2$

- Nullstellen des quadratischen Restpolynoms: $x_2 = -1$, $x_3 = 2$

⇒ Faktorzerlegung des Nenners $(x^3 - 3x - 2) = (x + 1)^2(x - 2)$

Ansatz für gebrochenen Anteil:

$$\frac{5x-1}{x^3-3x-2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} = \frac{A(x+1)(x-2)+B(x-2)+C(x+1)^2}{(x+1)^2(x-2)}$$

$$\text{Koeffizientenvergleich: } 5x - 1 = (A + C)x^2 + (-A + B + 2C)x + (-2A - 2B + C)$$

$$\begin{aligned} \Rightarrow \quad x^0 &: -2A - 2B + C = -1 \\ x^1 &: -A + B + 2C = 5 \quad \Rightarrow \quad A = -1, B = 2, C = 1 \\ x^2 &: A + C = 0 \end{aligned}$$

$$\text{Integration der Partialbrüche: } \int \left(-\frac{1}{x+1} + \frac{2}{(x+1)^2} + \frac{1}{x-2} \right) dx = -\ln |x+1| - \frac{2}{x+1} + \ln |x-2|$$

Zusammenfassung:

$$\int \frac{-2x^5+9x^3+4x^2-4x-7}{x^3-3x-2} dx = -\frac{2}{3}x^3 + 3x - \ln |x+1| - \frac{2}{x+1} + \ln |x-2|$$

$$\text{g) } \int \frac{2x^2+9x+12}{x^2+6x+10} dx = \int \left(2 - \frac{3x+8}{x^2+6x+10} \right) dx = 2x - \frac{3}{2} \ln(x^2+6x+10) + \arctan(x+3)$$

$$\begin{aligned} \text{h) } \int \frac{3x^4}{x^4+5x^2+4} dx &= \int \left(3 - \frac{15x^2+12}{x^4+5x^2+4} \right) dx = 3x - \int \frac{15x^2+12}{(x^2+4)(x^2+1)} dx = 3x - \int \left(\frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+1} \right) dx \\ &= 3x - \int \left(\frac{16}{x^2+4} + \frac{1}{x^2+1} \right) dx = 3x - 8 \arctan \frac{x}{2} + \arctan x \end{aligned}$$

$$\begin{aligned}
\text{i) } \int \frac{x^4+10x^2-5x-1}{x^3-x^2+4x-4} dx &= \int \left(x + 1 + \frac{7x^2-5x+3}{x^3-x^2+4x-4} \right) dx = \frac{x^2}{2} + x + \int \frac{7x^2-5x+3}{(x-1)(x^2+4)} dx \\
&= \frac{x^2}{2} + x + \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+4} \right) dx = \frac{x^2}{2} + x + \int \left(\frac{1}{x-1} + \frac{6x+1}{x^2+4} \right) dx \\
&= \frac{x^2}{2} + x + \ln|x-1| + 3 \ln(x^2+4) + \frac{1}{2} \arctan \frac{x}{2}
\end{aligned}$$

4. Lösungshinweise:

(Integrationskonstanten sind weggelassen.)

a) Substitution $\sqrt{x} = t$, $x = t^2$, $dx = 2t dt$ und partielle Integration:

$$\int e^{\sqrt{x}} dx = 2 \int te^t dt = 2(te^t - \int e^t dt) = 2(te^t - e^t) = 2e^{\sqrt{x}}(\sqrt{x} - 1)$$

b) Doppelwinkelformel: $1 - \cos 4x = 1 - \cos^2 2x + \sin^2 2x = 2 \sin^2 2x \Rightarrow$

$$\int \sqrt{1 - \cos 4x} dx = \int \sqrt{2 \sin^2 2x} dx = \sqrt{2} \int \sin 2x dx = -\frac{\sqrt{2}}{2} \cos 2x$$

c) Zweimal partiell integrieren:

$$\begin{aligned}
\int x^2 \cos 3x dx &= \frac{1}{3}x^2 \sin 3x - \frac{2}{3} \int x \sin 3x dx, \quad \int x \sin 3x dx = -\frac{1}{3}x \cos 3x + \frac{1}{3} \int \cos 3x dx \\
\Rightarrow \int x^2 \cos 3x dx &= \left(\frac{x^2}{3} - \frac{2}{27} \right) \sin 3x + \frac{2}{9}x \cos 3x.
\end{aligned}$$

d) Zweimal partiell integrieren:

$$\begin{aligned}
\int e^x \sin x dx &= e^x(-\cos x) + \int e^x \cos x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx \\
\Rightarrow 2 \int e^x \sin x dx &= e^x \sin x - e^x \cos x \Rightarrow \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x).
\end{aligned}$$

e) Substitution $\cos x = t$, $-\sin x dx = dt$:

$$\begin{aligned}
\int \frac{\sin^3 x}{\cos^4 x} dx &= \int \frac{\sin^2 x}{\cos^4 x} \sin x dx = \int \frac{1-\cos^2 x}{\cos^4 x} \sin x dx = -\int \frac{1-t^2}{t^4} dt = \int \frac{-1}{t^4} dt + \int \frac{1}{t^2} dt \\
&= \frac{1}{3t^3} - \frac{1}{t} = \frac{1}{3 \cos^3 x} - \frac{1}{\cos x}.
\end{aligned}$$

f) Substitution $e^x = t$, $dx = \frac{dt}{t}$:

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{1}{t + \frac{1}{t}} \frac{dt}{t} = \int \frac{1}{t^2 + 1} dt = \arctan t = \arctan e^x.$$

g) Substitution $x = 2 \sin t$, $dx = 2 \cos t dt$, Doppelwinkelformeln:

$$\begin{aligned}
\int \sqrt{4 - x^2} dx &= \int \sqrt{4 - 4 \sin^2 t} 2 \cos t dt = \int 4 \cos^2 t dt = \int 2(1 + \cos 2t) dt \\
&= 2t + \sin 2t = 2t + 2 \sin t \cos t = 2t + 2 \sin t \sqrt{1 - \sin^2 t} \\
&= 2 \arcsin \frac{x}{2} + \frac{x}{2} \sqrt{4 - x^2}.
\end{aligned}$$

h) Substitution $x = \sinh t$, $dx = \cosh t dt$, Beziehungen Hyperbelfunktionen:

$$\begin{aligned}
\int \frac{2x^2}{\sqrt{1+x^2}} dx &= \int \frac{2 \sinh^2 t}{\sqrt{1+\sinh^2 t}} \cosh t dt = \int 2 \sinh^2 t dt = \int (\cosh 2t - 1) dt \\
&= \frac{1}{2} \sinh 2t - t = \sinh t \cosh t - t = \sinh t \sqrt{1 + \sinh^2 t} - t \\
&= x \sqrt{1 + x^2} - \operatorname{arsinh} x.
\end{aligned}$$

i) Nach Umformung Substitution $\frac{x-2}{2} = t$, $\frac{1}{2} dx = dt$:

$$\int \frac{dx}{\sqrt{x^2 - 4x + 8}} = \int \frac{dx}{\sqrt{(x-2)^2 + 4}} = \int \frac{dx}{2\sqrt{\left(\frac{x-2}{2}\right)^2 + 1}} = \int \frac{dt}{\sqrt{t^2 + 1}} = \operatorname{arsinh} t = \operatorname{arsinh} \left(\frac{x-2}{2} \right).$$

j) Substitution $e^x = t$, $dx = \frac{dt}{t}$:

$$\begin{aligned} \int \frac{e^{3x}}{e^x+1} dx &= \int \frac{t^3}{t+1} \frac{dt}{t} = \int \frac{t^2}{t+1} dt = \int \left(t - 1 + \frac{1}{t+1} \right) dt = \frac{t^2}{2} - t + \ln|t+1| \\ &= \frac{e^{2x}}{2} - e^x + \ln(e^x + 1). \end{aligned}$$

k) Substitution $\tan \frac{x}{2} = t$, $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$

$$\int \frac{dx}{\sin x} = \int \frac{1+t^2}{2t} \frac{2dt}{1+t^2} = \int \frac{dt}{t} = \ln|t| = \ln|\tan \frac{x}{2}|.$$

l) Substitution $\tan \frac{x}{2} = t$, $\cot x = \frac{1-t^2}{2t}$, $dx = \frac{2dt}{1+t^2}$ und Partialbruchzerlegung:

$$\begin{aligned} \int (1 - \cot^2 x) dx &= \int \left[1 - \left(\frac{1-t^2}{2t} \right)^2 \right] \frac{2dt}{1+t^2} = -\frac{1}{2} \int \frac{t^4 - 6t^2 + 1}{t^2(1+t^2)} dt = -\frac{1}{2} \int \left(1 + \frac{-7t^2+1}{t^2(1+t^2)} \right) dt \\ &= -\frac{1}{2}t - \frac{1}{2} \int \left(\frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{1+t^2} \right) dt = -\frac{1}{2}t - \frac{1}{2} \int \left(\frac{1}{t^2} + \frac{-8}{1+t^2} \right) dt \\ &= -\frac{1}{2}t + \frac{1}{2t} + 4 \arctan t = -\frac{1}{2} \tan \frac{x}{2} + \frac{1}{2 \tan \frac{x}{2}} + 2x = \frac{-\tan^2 \frac{x}{2} + 1}{2 \tan \frac{x}{2}} + 2x \\ &= \frac{1}{\tan x} + 2x = \cot x + 2x. \end{aligned}$$

Es geht aber auch einfacher! Wenn man sich in den Ableitungen der elementaren Funktionen auskennt, dann weiss man, dass $(\cot x)' = -1 - \cot^2 x$. Bis auf eine additive Konstante ist das (fast) der Integrand. Durch einfache Umformung bekommt man deshalb

$$\int (1 - \cot^2 x) dx = \int (2 - 1 - \cot^2 x) dx = \int 2 dx + \int (-1 - \cot^2 x) dx = 2x + \cot x.$$

5. Lösungshinweise:

a) Substitution $\tan x = t$, $dx = \frac{dt}{1+t^2} \Rightarrow$

$$\begin{aligned} \int \tan^3 x dx &= \int \frac{t^3}{1+t^2} dt = \int \left(t - \frac{t}{1+t^2} \right) dt = \int t dt - \frac{1}{2} \int \frac{2t}{1+t^2} dt = \frac{t^2}{2} - \frac{1}{2} \ln(1+t^2) + C \\ &= \frac{1}{2} \tan^2 x - \frac{1}{2} \ln(1 + \tan^2 x) + C \end{aligned}$$

b) Substitution $\cos x = t$, $\sin x = \sqrt{1-t^2}$, $dx = -\frac{dt}{\sqrt{1-t^2}} \Rightarrow$

$$\int \tan^3 x dx = \int \left(\frac{\sqrt{1-t^2}}{t^3} \right)^3 \left(\frac{dt}{\sqrt{1-t^2}} \right) = \int \frac{dt}{t} - \int \frac{dt}{t^3} = \ln|t| + \frac{1}{2t^2} + \tilde{C} = \ln|\cos x| + \frac{1}{2\cos^2 x} + \tilde{C}$$

Wegen $\frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \tan^2 x + 1$ gilt

$$\begin{aligned} \frac{1}{2} \tan^2 x - \frac{1}{2} \ln(1 + \tan^2 x) + C &= \frac{1}{2} \left(\frac{1}{\cos^2 x} - 1 \right) - \frac{1}{2} \ln \frac{1}{\cos^2 x} + C = \frac{1}{2\cos^2 x} - \frac{1}{2} + \ln \sqrt{\cos^2 x} + C \\ &= \frac{1}{2\cos^2 x} + \ln|\cos x| + C - \frac{1}{2}, \end{aligned}$$

die gefundenen Lösungen unterscheiden sich also nur um eine additive Konstante und gehen für $\tilde{C} = C - \frac{1}{2}$ ineinander über.

6. Lösungshinweise:

a) Substitution $1 + \sqrt{x} = t$, $\frac{1}{2\sqrt{x}} dx = dt$, $dx = 2(t-1) dt$:

$$\int_1^9 \frac{\sqrt{x}}{1+\sqrt{x}} dx = \int_2^4 \frac{t-1}{t} 2(t-1) dt = 2 \int_2^4 \left(t - 2 + \frac{1}{t} \right) dt = 2 \left[\frac{t^2}{2} - 2t + \ln|t| \right]_2^4 = 4 + 2 \ln 2.$$

b) Uneigentliches Integral, Nenner unbeschränkt für $x \rightarrow 0$:

Unbestimmtes Integral (Zähler ist Ableitung des Nenners): $\int \frac{e^x}{e^x-1} dx = \ln|e^x - 1|$

Unerlaubte Anwendung des Hauptsatzes der Integralrechnung führt auf den **Fehler**:

$$\int_{-1}^1 \frac{e^x}{e^x-1} dx = \left[\ln|e^x - 1| \right]_{-1}^1 = \ln|e - 1| - \ln\left|\frac{1}{e} - 1\right| = \frac{\ln(e-1)}{\ln(1-\frac{1}{e})} = \frac{\ln(e-1)}{\ln(\frac{e-1}{e})} = \frac{\ln(e-1)}{\ln(e-1)-1} .$$

Erforderlich ist jedoch die Grenzwertbetrachtung (ε, μ unabhängig voneinander !)

$$\begin{aligned} \int_{-1}^1 \frac{e^x}{e^x-1} dx &= \lim_{\varepsilon \rightarrow +0} \int_{-1}^{-\varepsilon} \frac{e^x}{e^x-1} dx + \lim_{\mu \rightarrow +0} \int_{\mu}^1 \frac{e^x}{e^x-1} dx \\ &= \lim_{\varepsilon \rightarrow +0} (\ln|\frac{1}{e^\varepsilon} - 1| - \ln|\frac{1}{e} - 1|) + \lim_{\mu \rightarrow +0} (\ln|e - 1| - \ln|e^\mu - 1|) \\ &= \frac{\ln(e-1)}{\ln(e-1)-1} + \lim_{\varepsilon \rightarrow +0} \ln|\frac{1}{e^\varepsilon} - 1| - \lim_{\mu \rightarrow +0} \ln|e^\mu - 1| = (-\infty + \infty), \quad \text{divergent.} \end{aligned}$$

c) Substitution $\sqrt{x^2 - 1} = t$, $\frac{x}{\sqrt{x^2-1}} dx = dt$, $x dx = t dt$, partielle Integration:

$$\int_1^2 x e^{\sqrt{x^2-1}} dx = \int_0^{\sqrt{3}} t e^t dt = [t e^t]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} e^t dt = e^{\sqrt{3}}(\sqrt{3} - 1) + 1 .$$

7. Lösungshinweise:

a) $f(x) = x^3 - 7x^2 + 10x = 0$ für $x = 0, x = 2, x = 5$, $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty \Rightarrow$

$$A = \left| \int_0^2 (x^3 - 7x^2 + 10x) dx \right| + \left| \int_2^5 (x^3 - 7x^2 + 10x) dx \right| = \frac{16}{3} + \frac{189}{12} = \frac{253}{12} .$$

b) $y = 2\sqrt{x}$, definiert für $x \geq 0$, $y = \sqrt{1-x}$, definiert für $x \leq 1$, sind nichtnegativ und schneiden sich bei $x = \frac{1}{5}$. Die von beiden Kurven (und der x -Achse) eingeschlossene Fläche hat den Inhalt:

$$A = 2 \int_0^{\frac{1}{5}} \sqrt{x} dx + \int_{\frac{1}{5}}^1 \sqrt{1-x} dx = 2 \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^{\frac{1}{5}} - \left[\frac{2}{3} (1-x)^{\frac{3}{2}} \right]_{\frac{1}{5}}^1 = \frac{4}{15} \sqrt{5} .$$

8. Lösungshinweise:

$$\int_0^{x_0} e^{\frac{x}{2}} dx = \int_{x_0}^4 e^{\frac{x}{2}} dx, \quad 2 \left[e^{\frac{x}{2}} \right]_0^{x_0} = 2 \left[e^{\frac{x}{2}} \right]_{x_0}^4, \quad e^{\frac{x_0}{2}} - 1 = e^2 - e^{\frac{x_0}{2}}, \quad e^{\frac{x_0}{2}} = \frac{e^2+1}{2}, \quad x_0 = 2 \ln\left(\frac{e^2+1}{2}\right) .$$

9. Lösungshinweise:

Der Flächeninhalt a ist gerade das Doppelte der Fläche, die begrenzt wird von der x -Achse, der Geraden $y = \frac{y_0}{x_0} x$ und dem Hyperbelbogen. Es gilt also $a = 2 \int_0^{y_0} \left(\sqrt{1+y^2} - \frac{x_0}{y_0} y \right) dy$.

In einer Nebenrechnung findet man: (partielle Integration $u = \sqrt{1+y^2}$, $v' = 1$)

$$\begin{aligned} \int \sqrt{1+y^2} dy &= \sqrt{1+y^2} \cdot y - \int \frac{1}{2} \frac{2y}{\sqrt{1+y^2}} \cdot y dy = y\sqrt{1+y^2} - \int \frac{y^2+1-1}{\sqrt{1+y^2}} dy \\ &= y\sqrt{1+y^2} - \int \sqrt{1+y^2} dy + \int \frac{1}{\sqrt{1+y^2}} dy \end{aligned}$$

$$\Rightarrow \quad 2 \int \sqrt{1+y^2} dy = y\sqrt{1+y^2} + \operatorname{arsinh} y, \quad \boxed{\int \sqrt{1+y^2} dy = \frac{1}{2} \left(y\sqrt{1+y^2} + \operatorname{arsinh} y \right)} \quad (*)$$

Lösungsvariante: Substitution $y = \sinh t$, $\sqrt{1+y^2} = \sqrt{1+\sinh^2 t} = \cosh t$, $dy = \cosh t dt$ führt auf das Integral $\int \cosh^2 t dt = \frac{1}{2}(\sinh t \cosh t + t)$ (gelöst durch partielle Integration in einer früheren Aufgabe). Rücksubstitution ($t = \operatorname{arsinh} y$) ergibt wieder (*).

Damit erhält man unter Beachtung von $x_0 = \sqrt{1+y_0^2}$, daß

$$a = 2 \int_0^{y_0} \left(\sqrt{1+y^2} - \frac{x_0}{y_0} y \right) dy = y\sqrt{1+y^2} + \operatorname{arsinh} y - \frac{x_0}{y_0} y^2 \Big|_{y=0}^{y_0} \\ = y_0 \sqrt{1+y_0^2} + \operatorname{arsinh} y_0 - x_0 y_0 = \operatorname{arsinh} y_0 .$$

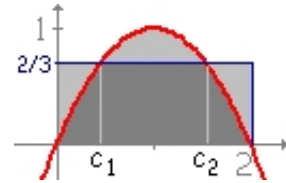
Folglich gilt $y_0 = \sinh a$ und also auch $x_0 = \sqrt{1+\sinh^2 a} = \cosh a$.

10. Lösungshinweise:

Mittelwertsatz: $\int_a^b f(x) dx = f(c)(b-a)$

$$\int_0^2 1 - (x-1)^2 dx = x - \frac{1}{3}(x-1)^3 \Big|_0^2 = \frac{4}{3} \stackrel{!}{=} f(c)(2-0)$$

$$\Rightarrow 1 - (c-1)^2 = \frac{2}{3}, \quad c_{1,2} = 1 \pm \frac{1}{3}\sqrt{3}$$



11. Lösungshinweise:

Bogenlänge für ebene Kurven in kartesischen Koordinaten $y = f(x)$, $a \leq x \leq b$:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx .$$

a) $y = \frac{x^3}{12} + \frac{1}{x}$, $x \in [2, 4]$, $s = \int_2^4 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2} dx = \int_2^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_2^4 = \frac{59}{12}$.

b) $y = \ln \sin x$, $x \in [\frac{\pi}{3}, \frac{\pi}{2}]$; Substitution $\tan \frac{x}{2} = t$, $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$,

$$s = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} dx = \int_{\frac{\tan \frac{\pi}{4}}{\tan \frac{\pi}{6}}}^{\frac{\tan \frac{\pi}{4}}{\tan \frac{\pi}{2}}} \frac{1+t^2}{2t} \frac{2dt}{1+t^2} = \int_{\frac{\sqrt{3}}{3}}^1 \frac{dt}{t} = \left[\ln |t| \right]_{\frac{\sqrt{3}}{3}}^1 = \frac{1}{2} \ln 3 .$$

c) $y = \operatorname{arcsin} e^{-x}$, $x \in [\ln 2; 1]$; Substitution $e^x = t$,

$$s = \int_{\ln 2}^1 \sqrt{1 + \left(\frac{-e^{-x}}{\sqrt{1-(e^{-x})^2}}\right)^2} dx = \int_{\ln 2}^1 \sqrt{\frac{1-e^{-2x}+e^{-2x}}{1-e^{-2x}}} dx = \int_{\ln 2}^1 \sqrt{\frac{1}{1-e^{-2x}}} dx \\ = \int_{\ln 2}^1 \sqrt{\frac{e^{2x}}{e^{2x}-1}} dx = \int_{\ln 2}^1 \frac{e^x}{\sqrt{e^{2x}-1}} dx = \int_{\ln 2}^e \frac{e^x}{\sqrt{e^{2x}-1}} dx = \int_2^e \frac{1}{\sqrt{t^2-1}} dt \\ = \left[\operatorname{arccosh} t \right]_2^e = \left[\ln(t + \sqrt{t^2-1}) \right]_2^e = \ln(e + \sqrt{e^2-1}) - \ln(2 + \sqrt{3}) .$$

Bogenlänge für ebene Kurven in Parameterdarstellung $x = x(t)$, $y = y(t)$, $a \leq t \leq b$:

$$s = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

d) $x = t^2$, $y = 2t^3$, $0 \leq t \leq \frac{\sqrt{3}}{3}$: $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{4t^2(1+9t^2)} = 2t\sqrt{1+9t^2}$,

$$s = 2 \int_{t=0}^{\frac{\sqrt{3}}{3}} t \sqrt{1+9t^2} dt = \frac{2}{27} [(1+9t^2)^{\frac{\sqrt{3}}{3}}]_{t=0}^{\frac{\sqrt{3}}{3}} = \frac{14}{27} \quad .$$

e) $x = \frac{1}{2}t^2$, $y = t \cosh t - \sinh t$, $0 \leq t \leq 1$: $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{t^2 + t^2 \sinh^2 t} = t \cosh t$,

$$s = \int_{t=0}^1 t \cosh t dt = [t \sinh t - \cosh t]_{t=0}^1 \quad (\text{Partielle Integration})$$

$$= \sinh 1 - \cosh 1 + \cosh 0 = \frac{e^1 - e^{-1}}{2} - \frac{e^1 + e^{-1}}{2} + 1 = 1 - \frac{1}{e} \quad .$$

12. Lösungshinweise:

$$(a) L_1 = \int_0^{x_1} \sqrt{1+(y')^2} dx = \int_0^{x_1} \sqrt{1+x} dx = \frac{2}{3} (1+x)^{\frac{3}{2}} \Big|_0^{x_1}$$

$$= \frac{2}{3} \left[(1+x_1)^{\frac{3}{2}} - 1 \right] \stackrel{!}{=} \frac{14}{3}$$

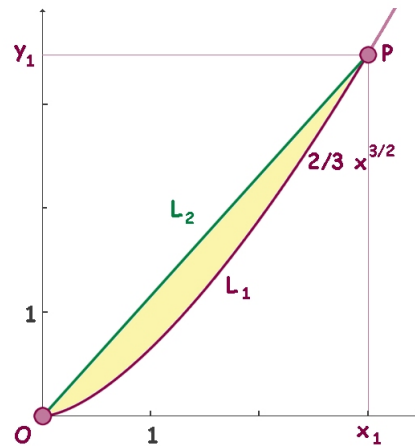
$$\Rightarrow x_1 = \left(\frac{14}{3} \cdot \frac{3}{2} + 1 \right)^{\frac{2}{3}} - 1 = \sqrt[3]{8^2} - 1 = 3$$

$$\Rightarrow y_1 = \frac{2}{3} \left(3^{\frac{3}{2}} \right) = \frac{2}{3} \sqrt{3} \cdot 3 = 2\sqrt{3}$$

$$L_2 = \sqrt{x_1^2 + y_1^2} = \sqrt{9 + 12} = \sqrt{21}$$

$$L = L_1 + L_2 = \frac{14}{3} + \sqrt{21}$$

$$(b) A = \int_0^{x_1} \left(\frac{y_1}{x_1} x - \frac{2}{3} x^{\frac{3}{2}} \right) dx = \int_0^3 \left(\frac{2\sqrt{3}}{3} x - \frac{2}{3} x^{\frac{3}{2}} \right) dx = \frac{\sqrt{3}}{3} x^2 - \frac{14}{5} x^{\frac{5}{2}} \Big|_0^3 = \frac{3}{5} \sqrt{3}$$



13. Lösungshinweise:

$$(a) V = \pi \int_0^{\ln 2} f(x)^2 dx = 2\pi \int_0^{\ln 2} e^{2x} dx = \frac{\pi}{2} [e^{2x}]_0^{\ln 2} = \frac{\pi}{2} (e^{2 \ln 2} - e^0) = \frac{3}{2} \pi$$

$$M = 2\pi \int_0^{\ln 2} f(x) \sqrt{1+[f'(x)]^2} dx = 2\pi \int_0^{\ln 2} e^x \sqrt{1+e^{2x}} dx$$

Substitution: $e^x = t$, $x = \ln t$, $e^x dx = dt$, $t_1 = e^0 = 1$, $t_2 = e^{\ln 2} = 2$

$$M = 2\pi \int_1^2 \sqrt{1+t^2} dt$$

1. Variante zur Lösung des Integrals $\int \sqrt{1+t^2} dt$

Substitution $t = \sinh u$, $u = \operatorname{arsinh} t = \ln(t + \sqrt{t^2 + 1})$, $dt = \cosh u du$,
 $u_1 = \operatorname{arsinh} 1 = \ln(1 + \sqrt{2})$, $u_2 = \operatorname{arsinh} 2 = \ln(2 + \sqrt{5})$

und Anwendung von Beziehungen zwischen Hyperbelfunktionen liefert

$$\begin{aligned}
M &= 2\pi \int_{u_1}^{u_2} \sqrt{1 + \sinh^2 u} \cosh u \, du = 2\pi \int_{u_1}^{u_2} \cosh^2 u \, du = 2\pi \int_{u_1}^{u_2} \frac{1}{2} (\cosh 2u + 1) \, du \\
&= \pi \left[\frac{1}{2} \sinh 2u + u \right]_{u_1}^{u_2} = \pi [\sinh u \cosh u + u]_{u_1}^{u_2} = \pi \left[\sinh u \sqrt{1 + \sinh^2 u} + u \right]_{u_1}^{u_2} \\
&= \pi \left[t\sqrt{1+t^2} + \ln(t + \sqrt{t^2+1}) \right]_{t_1=1}^{t_2=2} \quad (\star) \\
&= \pi \left(2\sqrt{5} + \ln(2 + \sqrt{5}) - \sqrt{2} - \ln(1 + \sqrt{2}) \right) = \pi \left(2\sqrt{5} - \sqrt{2} + \ln \frac{2+\sqrt{5}}{1+\sqrt{2}} \right)
\end{aligned}$$

2. Variante zur Lösung des Integrals $\int \sqrt{1+t^2} \, dt$

(Standard-)Substitution für Integrale der Form $\int R(t; \sqrt{1+t^2}) \, dt$ (siehe Formelsammlungen)

$$t = \frac{v^2-1}{2v}, \quad v = t + \sqrt{t^2+1}, \quad \sqrt{t^2+1} = \frac{v^2+1}{2v}, \quad dt = \frac{v^2+1}{2v^2} \, dv, \quad v_1 = 1 + \sqrt{2}, \quad v_2 = 2 + \sqrt{5}$$

$$\begin{aligned}
M &= 2\pi \int_{v_1}^{v_2} \frac{v^2+1}{2v} \frac{v^2+1}{2v^2} \, dv = \frac{\pi}{2} \int_{v_1}^{v_2} \frac{v^4+2v^2+1}{v^3} \, dv = \frac{\pi}{2} \int_{v_1}^{v_2} \left(v + \frac{2}{v} + \frac{1}{v^3} \right) \, dv = \frac{\pi}{2} \left[\frac{v^2}{2} + 2 \ln |v| - \frac{1}{2v^2} \right]_{v_1}^{v_2} \\
&= \pi \left[\frac{(t+\sqrt{t^2+1})^2}{4} - \frac{1}{4(t+\sqrt{t^2+1})^2} + \ln(t + \sqrt{t^2+1}) \right]_{t_1=1}^{t_2=2} \quad (\star\star)
\end{aligned}$$

Vergleicht man (\star) mit $(\star\star)$, so besteht keine offensichtliche Übereinstimmung. Man rechnet jedoch (durch geeignetes Erweitern und Anwendung der binomischen Formeln) leicht nach, daß

$$\begin{aligned}
\frac{(t+\sqrt{t^2+1})^2}{4} - \frac{1}{4(t+\sqrt{t^2+1})^2} &= \frac{1}{4} \left\{ (t + \sqrt{t^2+1})^2 - \left(\frac{t-\sqrt{t^2+1}}{(t+\sqrt{t^2+1})(t-\sqrt{t^2+1})} \right)^2 \right\} \\
&= \frac{1}{4} \left\{ (t + \sqrt{t^2+1})^2 - \left(\frac{t-\sqrt{t^2+1}}{t^2-(t^2+1)} \right)^2 \right\} = \frac{1}{4} \left\{ (t + \sqrt{t^2+1})^2 - (t - \sqrt{t^2+1})^2 \right\} \\
&= \frac{1}{4} \left(t^2 + 2t\sqrt{t^2+1} + t^2 + 1 - (t^2 - 2t\sqrt{t^2+1} + t^2 + 1) \right) = t\sqrt{t^2+1}
\end{aligned}$$

3. Variante zur Lösung des Integrals $\int \sqrt{1+t^2} \, dt$, partielle Integration $u = \sqrt{1+t^2}$, $v' = 1$

$$\begin{aligned}
\int \sqrt{1+t^2} \, dt &= \sqrt{1+t^2} \cdot t - \int \frac{1}{2} \frac{2t}{\sqrt{1+t^2}} \cdot t \, dt = t\sqrt{1+t^2} - \int \frac{t^2+1-1}{\sqrt{1+t^2}} \, dt \\
&= t\sqrt{1+t^2} - \int \sqrt{1+t^2} \, dt + \int \frac{1}{\sqrt{1+t^2}} \, dt
\end{aligned}$$

$$\Rightarrow 2 \int \sqrt{1+t^2} \, dt = t\sqrt{1+t^2} + \operatorname{arsinh} t, \quad \int \sqrt{1+t^2} \, dt = \frac{1}{2} \left(t\sqrt{1+t^2} + \ln(1 + \sqrt{1+t^2}) \right)$$

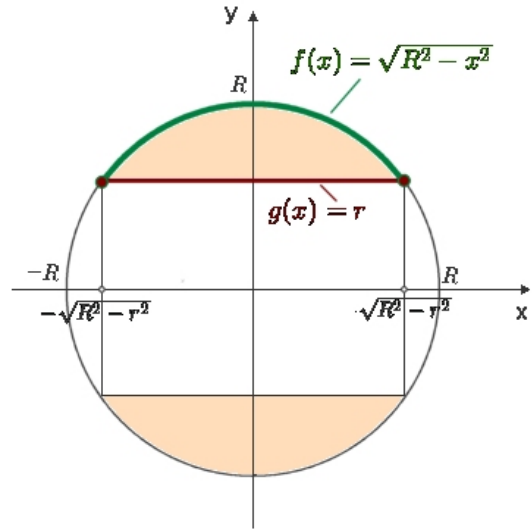
$$(b) \quad y = e^x, \quad 0 \leq x \leq \ln 2 \iff x = \ln y, \quad 1 \leq y \leq 2$$

$$\begin{aligned}
V &= \pi \int_1^2 g(y)^2 \, dy = \pi \int_1^2 (\ln y)^2 \, dy = \underset{(NR)}{\pi} [y(\ln y)^2 - 2y \ln y + 2y]_1^2 \\
&= \pi (2(\ln 2)^2 - 4 \ln 2 + 4 - (\ln 1)^2 - 2 \ln 1 - 2) = 2\pi ((\ln 2)^2 - 2 \ln 2 + 1) = 2\pi (1 - \ln 2)^2
\end{aligned}$$

14. Lösungshinweise:

Bei entsprechender Wahl des Koordinatensystems (Skizze) ergibt sich das gesuchte Volumen als Differenz der Volumina der für $\sqrt{R^2 - r^2} \leq x \leq -\sqrt{R^2 - r^2}$ von $f(x) = \sqrt{R^2 - x^2}$ (Kugelscheibe) bzw. $g(x) = r$ (Zylinder) erzeugten Rotationskörper:

$$\begin{aligned} V &= \pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} f(x)^2 dx - \pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} g(x)^2 dx \\ &= 2\pi \int_0^{\sqrt{R^2 - r^2}} (R^2 - x^2 - r^2) dx \\ &= 2\pi \left[(R^2 - r^2)x - \frac{1}{2}x^3 \right]_0^{\sqrt{R^2 - r^2}} = \frac{4}{3}\pi \left(\sqrt{R^2 - r^2} \right)^3 \end{aligned}$$



Anstelle mit dem 2. Integral im Ansatz läßt sich das entsprechende Zylindervolumen natürlich auch elementar berechnen als $\pi r^2 \cdot 2\sqrt{R^2 - r^2}$.

Die Oberfläche erhält man als Summe der Mantelflächen der Kugelscheibe („Außenfläche“) und des Zylinders („Innenfläche“):

$$\begin{aligned} A_O &= 2\pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} f(x) \sqrt{1 + [f'(x)]^2} dx + 2\pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} g(x) \sqrt{1 + [g'(x)]^2} dx \\ &= 2\pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left\{ \sqrt{R^2 - x^2} \sqrt{1 + \left[\frac{1}{2} \frac{1}{\sqrt{R^2 - x^2}} (-2x) \right]^2} + r \right\} dx \\ &= 2\pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left\{ \sqrt{R^2 - x^2} \sqrt{\frac{R^2 - x^2 + x^2}{R^2 - x^2}} + r \right\} dx \\ &= 2\pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} (R + r) dx = 4\pi (R + r) \sqrt{R^2 - r^2} \end{aligned}$$

15. Lösungshinweise:

$$\begin{aligned} \text{a) } \int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{a \rightarrow -\infty} \int_a^c \frac{dx}{1+x^2} + \lim_{b \rightarrow \infty} \int_c^b \frac{dx}{1+x^2} = - \lim_{a \rightarrow -\infty} \arctan a + \lim_{b \rightarrow \infty} \arctan b \\ &= - \left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) = \pi . \end{aligned}$$

$$\begin{aligned} \text{b) } \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 x} &= \lim_{a \rightarrow -\infty} \int_a^c \frac{dx}{\cosh^2 x} + \lim_{b \rightarrow \infty} \int_c^b \frac{dx}{\cosh^2 x} = - \lim_{a \rightarrow -\infty} \tanh a + \lim_{b \rightarrow \infty} \tanh b \\ &= -(-1) + 1 = 2 . \end{aligned}$$

$$\text{c) } \int_e^{\infty} \frac{dx}{x \ln x} = \lim_{b \rightarrow \infty} \int_e^b \frac{\frac{1}{x}}{\ln x} dx = \lim_{b \rightarrow \infty} \ln(\ln b) - \ln(\ln e) = \infty , \text{ bestimmt divergent.}$$

$$\text{d) } \int_0^{\infty} \cos(nx) dx = \lim_{b \rightarrow \infty} \int_0^b \cos(nx) dx = \frac{1}{n} \lim_{b \rightarrow \infty} \sin(nb) , \text{ unbestimmt divergent.}$$

$$\text{e) } \int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \lim_{t \rightarrow a-0} \frac{1}{a} \int_0^t \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \lim_{t \rightarrow a-0} \left[\arcsin \frac{x}{a}\right]_0^t = \lim_{t \rightarrow a-0} \arcsin \frac{t}{a} = \frac{\pi}{2} .$$

Lösungsvariante: Substitution $x = a \sin u$, $dx = a \cos u du$, $u = \arcsin \frac{x}{a}$:

$$\begin{aligned} \int_0^a \frac{dx}{\sqrt{a^2-x^2}} &= \lim_{t \rightarrow a-0} \int_0^t \frac{dx}{\sqrt{a^2-x^2}} = \lim_{t \rightarrow a-0} \int_{\arcsin 0}^{\arcsin \frac{t}{a}} \frac{a \cos u}{\sqrt{a^2-a^2 \sin^2 u}} du = \lim_{t \rightarrow a-0} \int_0^{\arcsin \frac{t}{a}} \frac{a \cos u}{a \sqrt{1-\sin^2 u}} du \\ &= \lim_{t \rightarrow a-0} \int_0^{\arcsin \frac{t}{a}} du = \lim_{t \rightarrow a-0} \arcsin \frac{t}{a} = \frac{\pi}{2} . \end{aligned}$$

$$\begin{aligned} \text{f) } \int_{-1}^1 \frac{dx}{\sqrt{|x|}} &= \lim_{\varepsilon \rightarrow +0} \int_{-1}^{-\varepsilon} \frac{dx}{\sqrt{-x}} + \lim_{\mu \rightarrow +0} \int_{\mu}^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \rightarrow +0} [-2\sqrt{-x}]_{-1}^{-\varepsilon} + \lim_{\mu \rightarrow +0} [2\sqrt{x}]_{\mu}^1 \\ &= \lim_{\varepsilon \rightarrow +0} -2\sqrt{\varepsilon} + 2 + 2 - 2 \lim_{\mu \rightarrow +0} \sqrt{\mu} = 4 . \end{aligned}$$

$$\begin{aligned} \text{g) } \int_0^{\frac{\pi}{2}} \frac{\cos x}{1-\sin x} dx &= - \lim_{t \rightarrow \frac{\pi}{2}-0} \int_0^t \frac{-\cos x}{1-\sin x} dx \left(= \int \frac{f'}{f} = \ln |f| \right) \\ &= - \lim_{t \rightarrow \frac{\pi}{2}-0} [\ln |1-\sin x|]_0^t = - \lim_{t \rightarrow \frac{\pi}{2}-0} (\ln |1-\sin t|) = \infty , \text{ divergent.} \end{aligned}$$

$$\begin{aligned} \text{h) } \int_0^9 \frac{dx}{\sqrt[3]{(x-1)^2}} &= \lim_{\varepsilon \rightarrow +0} \int_0^{1-\varepsilon} (1-x)^{-\frac{2}{3}} dx + \lim_{\mu \rightarrow +0} \int_{1+\mu}^9 (x-1)^{-\frac{2}{3}} dx \\ &= \lim_{\varepsilon \rightarrow +0} \left[-3\sqrt[3]{(1-x)}\right]_0^{1-\varepsilon} + \lim_{\mu \rightarrow +0} \left[3\sqrt[3]{(x-1)}\right]_{1+\mu}^9 \\ &= \lim_{\varepsilon \rightarrow +0} (-3\sqrt[3]{\varepsilon}) + 3 + 6 - \lim_{\mu \rightarrow +0} 3\sqrt[3]{\mu} = 9 . \end{aligned}$$